JEE Advanced (2023)

PAPER 1

Mathematics

General Instructions:

SECTION 1 (Maximum Marks: 12)

- This section contains **THREE (03)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+4 **ONLY** if (all) the correct option(s) is(are) chosen;
Partial Marks	:	+3 If all the four options are correct but **ONLY** three options are chosen;
Partial Marks	:	+2 If three or more options are correct but **ONLY** two options are chosen, both of which are correct;
Partial Marks	:	+1 If two or more options are correct but **ONLY** one option is chosen and it is a correct option;
Zero Marks	:	0 If none of the options is chosen (i.e., the question is unanswered);
Negative Marks	:	–2 In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
choosing ONLY (A), (B) and (D) will get +4 marks;
choosing ONLY (A) and (B) will get +2 marks;
choosing ONLY (A) and (D) will get +2 marks;
choosing ONLY (B) and (D) will get +2 marks;
choosing ONLY (A) will get +1 mark;
choosing ONLY (B) will get +1 mark;
choosing ONLY (D) will get +1 mark;
choosing no option (i.e., the question is unanswered) will get 0 marks; and
choosing any other combination of options will get –2 marks.

Q. 1. Let $S = (0, 1) \cup (1, 2) \cup (3, 4)$ and $T = \{0, 1, 2, 3\}$. Then which of the following statements is (are) true?

(A) There are infinitely many functions from S to T.

(B) There are infinitely many strictly increasing functions from S to T.

(C) The number of continuous functions from S to T is at most 120.

(D) Every continuous function from S to T is differentiable.

Q. 2. Let T_1 and T_2 be two distinct common tangents to the ellipse $E: \frac{x^2}{6} + \frac{y^2}{3} = 1$ and the parabola $P: y^2 = 12x$. Suppose that the tangent T_1 touches P and E at the points A_1 and A_2, respectively and the tangent T_2 touches P and E at the points A_4 and A_3, respectively. Then which of the following statements is (are) true?

(A) The area of the quadrilateral $A_1 A_2 A_3 A_4$ is 35 square units.

(B) The area of the quadrilateral $A_1 A_2 A_3 A_4$ is 36 square units.

(C) The tangents T_1 and T_2 meet the x-axis at the points (–3, 0).

(D) The tangents T_1 and T_2 meet the x-axis at the points (–6, 0).

Q. 3. Let $f : [0, 1] \rightarrow [0, 1]$ be the function defined by $f(x) = \frac{x^3}{3} - x^2 + \frac{5}{9}x + \frac{17}{36}$. Consider the square region $S = [0, 1] \times [0, 1]$. Let $G = \{(x, y) \in S : y > f(x)\}$ be called the green region and $R = \{(x, y) \in S : y < f(x)\}$ be called the red region. Let $L_h = \{(x, h) \in S : x \in [0, 1]\}$ be the horizontal line drawn at a height $h \in [0, 1]$. Then which of the following statements is (are) true?

(A) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the green region above the line L_h equals the area of the green region below the line L_h.

(B) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the red region above the line L_h equals the area of the red region below the line L_h.

(C) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the green region above the line L_h equals the area of the red region below the line L_h.

(D) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the red region above the line L_h equals the area of the green region below the line L_h.

General Instructions:

SECTION 2 (Maximum Marks: 12)

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If **ONLY** the correct option is chosen;
Zero Marks : 0 If none of the options is chosen (i.e., the question is unanswered);
Negative Marks : −1 In all other cases.

Q. 4. Let $f : (0, 1) \to \mathbb{R}$ be the function defined as $f(x) = \sqrt{n}$ if $x \in \left[\frac{1}{n+1}, \frac{1}{n}\right)$ where $n \in \mathbb{N}$. Let $g : (0, 1) \to \mathbb{R}$ be a function such that $\int_{x^2}^{x} \sqrt{\frac{1-t}{t}}\, dt < g(x) < 2\sqrt{x}$ for all $x \in (0, 1)$. Then $\lim_{x \to 0} f(x)g(x)$

(A) does NOT exists **(B)** is equal to 1
(C) is equal to 2 **(D)** is equal to 3

Q. 5. Let Q be the cube with the set of vertices $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1, x_2, x_3 \in \{0, 1\}\}$. Let F be the set of all twelve lines containing the diagonals of the six faces of the cube Q. Let S be the set of all four lines containing the main diagonals of the cube Q; for instance, the line passing through the vertices (0, 0, 0) and (1, 1, 1) is in S. For lines l_1 and l_2, let $d(l_1, l_2)$ denote the shortest distance between them. Then the maximum value of $d(l_1, l_2)$, as l_1 varies over F and l_2 varies over S, is

(A) $\frac{1}{\sqrt{6}}$ **(B)** $\frac{1}{\sqrt{8}}$
(C) $\frac{1}{\sqrt{3}}$ **(D)** $\frac{1}{\sqrt{12}}$

Q. 6. Let $X = \left\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : \frac{x^2}{8} + \frac{y^2}{20} < 1 \text{ and } y^2 < 5x\right\}$. Three distinct points P, Q and R are randomly chosen from X. Then the probability that P, Q and R from a triangle whose area is a positive integer, is

(A) $\frac{71}{220}$ **(B)** $\frac{73}{220}$
(C) $\frac{79}{220}$ **(D)** $\frac{83}{220}$

Q. 7. Let P be a point on the parabola $y^2 = 4ax$, where $a > 0$. The normal to the parabola at P meets the x-axis at a point Q. The area of the triangle PFQ, where F is the focus of the parabola, is 120. If the slope m of the normal and a are both positive integers, then the pair (a, m) is

(A) (2, 3) **(B)** (1, 3)
(C) (2, 4) **(D)** (3, 4)

General Instructions:

SECTION 3 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER**.
- For each question, enter the correct integer corresponding to the answer using the mouse and the onscreen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If **ONLY** the correct integer is entered;
Zero Marks : 0 In all other cases.

Q. 8. Let $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, for $x \in \mathbb{R}$. Then the number of real solutions of the equation

$\sqrt{1+\cos(2x)} = \sqrt{2}\tan^{-1}(\tan x)$ in the set $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is equal to

Q. 9. Let $n \geq 2$ be a natural number and $f : [0, 1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} n(1-2nx) & \text{if } 0 \leq x \leq \frac{1}{2n} \\ 2n(2nx-1) & \text{if } \frac{1}{2n} \leq x \leq \frac{3}{4n} \\ 4n(1-nx) & \text{if } \frac{3}{4n} \leq x \leq \frac{1}{n} \\ \frac{n}{n-1}(nx-1) & \text{if } \frac{1}{n} \leq x \leq 1 \end{cases}$$

If n is such that the area of the region bounded by the curves $x = 0$, $x = 1$, $y = 0$ and $y = f(x)$ is 4, then the maximum value of the function f is

Q. 10. Let $\overbrace{75\cdots57}^{r}$ denote the $(r + 2)$ digit number where the first and the last digits are 7 and the remaining r digits are 5. Consider the sum $S = 77 + 757 + 7557 + \cdots + \overbrace{75\cdots57}^{98}$. If $S = \dfrac{\overbrace{75\cdots57}^{99} + m}{n}$, where m and n are natural numbers less than 3000, then the value of $m + n$ is

Q. 11. Let $A = \left\{ \dfrac{1967 + 1686i \sin\theta}{7 - 3i\cos\theta} : \theta \in \mathbb{R} \right\}$. If A contains exactly one positive integer n, then the value of n is

Q. 12. Let P be the plane $\sqrt{3}x + 2y + 3z = 16$ and let $S = \{\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k} : \alpha^2 + \beta^2 + \gamma^2 = 1$ and the distance of (α, β, γ) from the plane P is $\frac{7}{2}\}$.

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be three distinct vectors in S such that $|\vec{u} - \vec{v}| = |\vec{v} - \vec{w}| = |\vec{w} - \vec{u}|$. Let V be the volume of the parallelepiped determined by vectors $\vec{u}, \vec{v}$ and $\vec{w}$. Then the value of $\frac{80}{\sqrt{3}}V$ is

Q. 13. Let a and b be two non-zero real numbers. If the coefficient of x^5 in the expansion of $\left(ax^2 + \frac{70}{27bx}\right)^4$ is equal to the coefficient of x^{-5} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^7$, then the value of $2b$ is

***General Instructions*:**

SECTION 4 (Maximum Marks : 12)

- This section contains **FOUR (04)** Matching List Sets.
- Each set has **ONE** Multiple Choice Question.
- Each set has **TWO** lists: **List-I** and **List-II**.
- **List-I** has **Four** entries (P), (Q), (R) and (S) and **List-II** has **Five** entries (1), (2), (3), (4) and (5).
- **FOUR** options are given in each Multiple Choice Question based on **List-I** and **List-II** and **ONLY ONE** of these four options satisfies the condition asked in the Multiple Choice Question.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+3 **ONLY** if the option corresponding to the correct combination is chosen;
Zero Marks	:	0 If none none of the options is chosen (i.e., the question is unanswered);
Negative Marks	:	–1 In all other cases.

Q. 14. Let α, β and γ be real numbers. Consider the following system of linear equations

$x + 2y + z = 7$

$x + \alpha z = 11$

$2x - 3y + \beta z = \gamma$

Match each entry in **List-I** to the correct entries in **List-II.**

List-I	**List-II**
(P) If $\beta = \frac{1}{2}(7\alpha - 3)$ and $\gamma = 28$, then the system has	(1) a unique solution
(Q) If $\beta = \frac{1}{2}(7\alpha - 3)$ and $\gamma \neq 28$, then the system has	(2) no solution
(R) If $\beta \neq \frac{1}{2}(7\alpha - 3)$ where $\alpha = 1$ and $\gamma \neq 28$, then the system has	(3) infinitely many solution
(S) If $\beta \neq \frac{1}{2}(7\alpha - 3)$ where $\alpha = 1$ and $\gamma = 28$, then the system has	(4) $x = 11$, $y = -2$ and $z = 0$ as a solution
	(5) $x = -15$, $y = 4$ and $z = 0$ as a solution

The correct option is:

(A) (P) → (3) (Q) → (2) (R) → (1) (S) → (4)

(B) (P) → (3) (Q) → (2) (R) → (5) (S) → (4)

(C) (P) → (2) (Q) → (1) (R) → (4) (S) → (5)

(D) (P) → (2) (Q) → (1) (R) → (1) (S) → (3)

Q. 15. Consider the given data with frequency distribution

x_i	3	8	11	10	5	4
f_i	5	2	3	2	4	4

Match each entry in **List-I** to the correct entries in **List-II.**

List-I	**List-II**
(P) The mean of the above data is	(1) 2.5
(Q) The median of the above data is	(2) 5
(R) The mean deviation about the mean of the above data is	(3) 6
(S) The mean deviation about the median of the above data is	(4) 2.7
	(5) 2.4

The correct option is:

(A) (P) → (3) (Q) → (2) (R) → (4) (S) → (5)
(B) (P) → (3) (Q) → (2) (R) → (1) (S) → (5)
(C) (P) → (2) (Q) → (3) (R) → (4) (S) → (1)
(D) (P) → (3) (Q) → (3) (R) → (5) (S) → (5)

Q. 16. Let l_1 and l_2 be the lines $\vec{r}_1 = \lambda(\hat{i}+\hat{j}+\hat{k})$ and $\vec{r}_2 = (\hat{j}-\hat{k}) + \mu(\vec{i}+\hat{k})$, respectively. Let X be the set of all the planes H that contain the line l_1. For a plane H, let $d(H)$ denote the smallest possible distance between the points of l_2 and H. Let H_0 be a plane in X for which $d(H_0)$ is the maximum value of $d(H)$ as H varies over all planes in X.

Match each entry in **List-I** to the correct entries in **List-II.**

List-I	**List-II**
(P) The value of $d(H_0)$ is	(1) $\sqrt{3}$
(Q) The distance of the point (0, 1, 2) from H_0 is	(2) $\frac{1}{\sqrt{3}}$
(R) The distance of origin from H_0 is	(3) 0
(S) The distance of origin from the point of intersection of planes $y = z, x = 1$ and H_0 is	(4) $\sqrt{2}$
	(5) $\frac{1}{\sqrt{2}}$

The correct option is :

(A) (P) → (2) (Q) → (4) (R) → (5) (S) → (1)
(B) (P) → (5) (Q) → (4) (R) → (3) (S) → (1)
(C) (P) → (2) (Q) → (1) (R) → (3) (S) → (2)
(D) (P) → (5) (Q) → (1) (R) → (4) (S) → (2)

Q. 17. Let z be a complex number satisfying $|z|^3 + 2z^2 + 4\bar{z} - 8 = 0$, where $\bar{z}$ denotes the complex conjugate of z. Let the imaginary part of z be non-zero.

Match each entry in **List-I** to the correct entries in **List-II.**

List-I	**List-II**
(P) $|z|^2$ is equal to	(1) 12
(Q) $|z-\bar{z}|^2$ is equal to	(2) 4
(R) $|z|^2 + |z+\bar{z}|^2$ is equal to	(3) 8
(S) $|z+1|^2$ is equal to	(4) 10
	(5) 7

The correct option is:

(A) (P) → (1) (Q) → (3) (R) → (5) (S) → (4)
(B) (P) → (2) (Q) → (1) (R) → (3) (S) → (5)
(C) (P) → (2) (Q) → (4) (R) → (5) (S) → (1)
(D) (P) → (2) (Q) → (3) (R) → (5) (S) → (4)

ANSWER KEY

Q.No.	Answer key	Topic's name	Chapter's name
Section-I			
1	(A, C, D)	Number of Functions	Function, Continuity and Differentiability
2	(A, C)	Parabola and Ellipse	Ellipse
3	(B, C, D)	Area under two curves	Area under curves
4	(C)	Sandwich Theorem	Limits and Definite Integral
Section-II			
5	(A)	Shortest distance between two line	Three Dimensional
6	(B)	Probability based on geometrical problem	Probability, Parabola, Ellipse
7	(A)	Normal of parabola	Parabola
Section-III			
8	3	Number of solution of equation	Inverse Trigonometric Functions
9	8	Area under simple curves	Area under the curves
10	1219	Geometric Progression	Sequence and Series
11	281	Components of a complex number	Complex Number
12	45	Volume of Parallelpipped	Vector, Three Dimensional
13	3	General term	Binomial Theorem
Section-IV			
14	(A)	System of Linear Equations	Determinants
15	(A)	Mean, Median, Mean Deviation, Variance	Statistics
16	(B)	Point, Line and Plane	Three Dimensional
17	(B)	Modulus of complex number	Complex Number

JEE Advanced (2023)

PAPER 1

ANSWERS WITH EXPLANATIONS

Mathematics

1. Correct options are (A, C and D).

Given $S = (0, 1) \cup (1, 2) \cup (3, 4)$

$T = \{0, 1, 2, 3\}$

$\because$ For function $S \rightarrow T$, set S (domain) has infinite elements but set T (codomain) has only 4 elements.

$\therefore$ There are infinite functions from S to T and it is impossible to make a function which is strictly increasing from S to T.

$\therefore$ All functions must be many one.

$\therefore$ Option (A) is correct.

and option (B) is not correct.

According to domain it is possible to make a continuous function from S to T.

Total no of such functions are $= 4^3 = 64$.

$\therefore$ Option (C) is correct.

Also every continuous function is differentiable.

$\therefore$ Option (D) is correct.

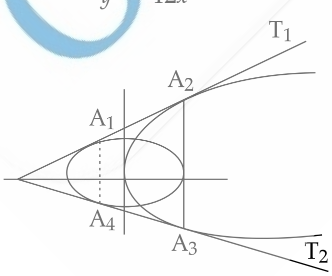
2. Correct options are (A and C).

$$E : \frac{x^2}{6} + \frac{y^2}{3} = 1$$

$$a^2 = 6$$

$$b^2 = 3$$

$$P : y^2 = 12x$$



Equation of tangent for ellipse

$$y = mx \pm \sqrt{6m^2 + 3} \quad \text{...(1)}$$

Equation of tangent for parabola

$$y = mx + \frac{a}{m}$$

$$\Rightarrow \quad y = mx + \frac{3}{m} \quad \text{...(2)}$$

By (1) and (2), we get

$$\frac{3}{m} = \pm\sqrt{6m^2 + 3}$$

and $$\frac{9}{m^2} = 6m^2 + 3$$

and $$9 = 6m^4 + 3m^2$$

$$2m^4 - m^2 + 3 = 0$$

$$(m^2 - 1)(2m^2 + 3) = 0$$

Or $$m^2 = 1$$

and $$m = \pm 1$$

$$2m^2 + 3 = 0 \text{ (which is not possible)}$$

$\therefore$ **Equation of tangents are**

$$y = x + 3 \text{ and } y = -x - 3$$

Now their point of intersection is $(-3, 0)$.

Equation of $A_1 A_4$

$$T = 0 \qquad \text{(Chord of contact for ellipse)}$$

$$\frac{x(-3)}{6} + \frac{y(0)}{4} = 1$$

$$x = -2$$

$\Rightarrow$ $A_1 = (-2, 1)$

and $A_4 = (-2, -1)$

Equation of $A_2 A_3$

$$T = 0 \qquad \text{(Chord of contact for parabola)}$$

$$y(0) = 12\left(\frac{x-3}{2}\right)$$

$\Rightarrow$ $x = 3$

$\Rightarrow$ $A_2 = (3, 6)$ and $A_3 = (3, -6)$

$\therefore$ Area of quadrilateral $A_1 A_2 A_3 A_4$

$$= \frac{1}{2} \times (2 + 12) \times 5 = 35 \text{ sq. units}$$

3. Correct options are (B, C and D).

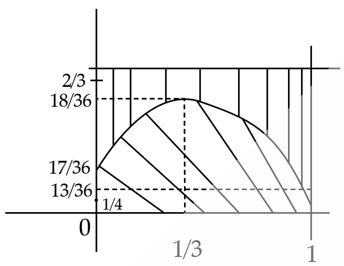
Given $f : [0, 1] \rightarrow [0, 1]$

$$f(x) = \frac{x^3}{3} - x^2 + \frac{5}{9}x + \frac{17}{36}$$

$$f'(x) = \frac{3x^2}{3} - 2x + \frac{5}{9}$$

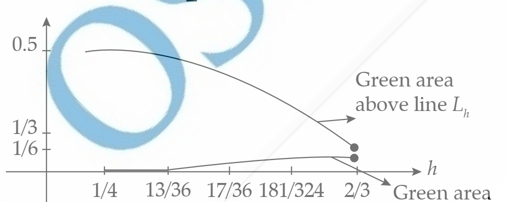
$$f'(x) = 0$$
$$9x^2 - 18x + 5 = 0$$
$$\Rightarrow 9x^2 - 15x - 3x + 5 = 0$$
$$\Rightarrow 3x(3x-5) - 1(3x-5) = 0$$
$$\Rightarrow (3x-5)(3x-1) = 0$$
$$\Rightarrow \quad x = \frac{1}{3} \text{ or } \frac{5}{3}$$
$$f''(x) = 2x - 2$$
$$f''\left(\frac{1}{3}\right) = \frac{2}{3} - 2 < 0 \text{ point of maxima}$$

Graph of $f(x)$



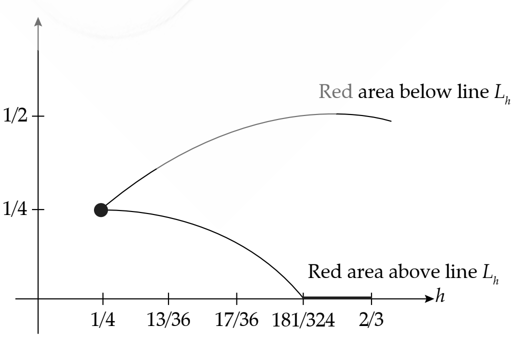
$$\text{Area}_{red} = \int_0^1 f(x)dx$$
$$= \left[\frac{x^4}{12} - \frac{x^3}{3} + \frac{5x^2}{18} + \frac{17x}{36}\right]_0^1$$
$$= \frac{1}{12} - \frac{1}{3} + \frac{5}{18} + \frac{17}{36}$$
$$= \frac{3 - 12 + 10 + 17}{36}$$
$$= \frac{18}{36} = \frac{1}{2} = 0.5$$
$$\therefore \quad (\text{Area})_{green} = 1 - \frac{1}{2} = 0.5$$

(A)



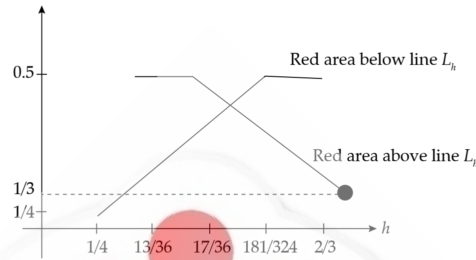
Both graph doesn't intersect

(B)



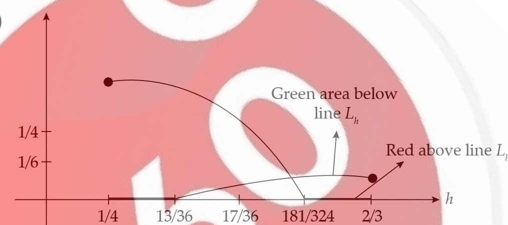
Both Graph intersect at $h = \frac{1}{4}$

(C)



Both Graph intersect.

(D)



Both Graph intersect.

4. Correct option is (C).

$f : (0, 1) \to R,$

$$f(x) = \sqrt{n}\ , x \in \left[\frac{1}{n+1}, \frac{1}{n}\right), n \in N$$

$g : (0, 1) \to R$ where

$$\int_{x^2}^{x} \sqrt{\frac{1-t}{t}}dt < g(x) < 2\sqrt{x}\ , x \in (0, 1)$$

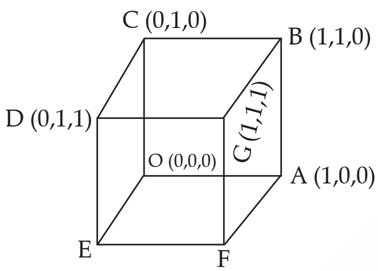
Now (According to the question)

$$\lim_{x\to\infty} f(x)\cdot g(x)$$
$$\Rightarrow \quad \text{Put} \quad x = \frac{1}{n}$$
$$\lim_{n\to\infty} f\left(\frac{1}{n}\right)g\left(\frac{1}{n}\right)$$
$$\lim_{n\to\infty} \sqrt{n-1}\int_{\frac{1}{n^2}}^{\frac{1}{n}} \sqrt{\frac{1-t}{t}}dt \le \lim_{n\to\infty} f\left(\frac{1}{n}\right)g\left(\frac{1}{n}\right)$$
$$\le \lim_{n\to\infty} \sqrt{n} - 1\frac{2}{\sqrt{n}}$$
$$\Rightarrow \quad \lim_{n\to\infty} \frac{\int_{\frac{1}{n^2}}^{\frac{1}{n}} \sqrt{\frac{1-t}{t}}\,dt}{\frac{1}{\sqrt{n-1}}} \le \lim_{n\to\infty} f\left(\frac{1}{n}\right)g\left(\frac{1}{n}\right) \le 2$$
$$\Rightarrow \quad \lim_{n\to\infty} \frac{\frac{-1}{n^2}\sqrt{n-1} + \frac{2}{n^3}\sqrt{n^2-1}}{\frac{1}{2(n-1)^{\frac{3}{2}}}}$$
$$\le \lim_{n\to\infty} f\left(\frac{1}{n}\right)g\left(\frac{1}{n}\right) \le 2$$

$$\therefore \quad \lim_{n\to\infty} \frac{2(n-1)^2}{n^2} - \frac{4(n-1)^{\frac{3}{2}}\sqrt{n^2-1}}{n^3} = 2$$

$$\therefore \lim_{n\to\infty} f\left(\frac{1}{n}\right) g\left(\frac{1}{n}\right) = 2 \text{ (Using Sandwich Theorem)}$$

5. Correct option is (A).



$$\overrightarrow{OG} = \hat{i} + \hat{j} + \hat{k} = \hat{b}_1$$

$$\overrightarrow{AC} = -\hat{i} + \hat{j} = \hat{b}_2$$

Equation of line OG

$$\Rightarrow \quad \frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

Equation of line AC

$$\Rightarrow \quad \frac{x-1}{-1} = \frac{y}{1} = \frac{z}{0}$$

$$\text{S.D.} = \frac{|(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1 \times \bar{b}_2)|}{|\bar{b}_1 \times \bar{b}_2|}$$

$$\bar{a}_2 - \bar{a}_1 = -\hat{i}$$

$$\bar{b}_1 \times \bar{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= \hat{i}(-1) - \hat{j}(+1) + \hat{k}(1+1)$$

$$= -\hat{i} - \hat{j} + 2\hat{k}$$

$$\text{S.D.} = \frac{|(-\hat{i}).(-\hat{i} - \hat{j} + 2\hat{k})|}{|-\hat{i}||-\hat{i} - \hat{j} + 2\hat{k}|}$$

$$= \frac{1}{1\sqrt{1+1+4}} = \frac{1}{\sqrt{6}}$$

6. Correct option is (B).

$$\frac{x^2}{8} + \frac{y^2}{20} < 1 \text{ and } y^2 < 5x$$

Let $\quad \dfrac{x^2}{8} + \dfrac{y^2}{20} = 1 \qquad$ ...(1)

and $\quad y^2 = 5x \qquad$ ...(2)

On solving (1) and (2), we get

$$\frac{x^2}{8} + \frac{5x}{20} = 1$$

$$\frac{x^2}{8} + \frac{x}{4} = 1$$

$$x^2 + 2x = 8$$

$$\Rightarrow \quad x^2 + 2x - 8 = 0$$

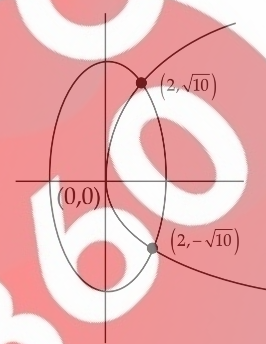
$$\Rightarrow \quad (x+4)(x-2) = 0$$

$$\Rightarrow \quad x = -4, 2$$

$$\Rightarrow \quad x = 2 \text{ (}-4 \text{ is not possible)}$$

$$\Rightarrow \quad y^2 = 10$$

$$y = \pm\sqrt{10}$$



$X = \{(1, 1), (1, 0), (1, -1), (1, 2), (1, -2), (2, 1), (2 -1), (2, 3), (2, 3), (2, -3), (2, -2), (2, 2), (2, 0)\}$

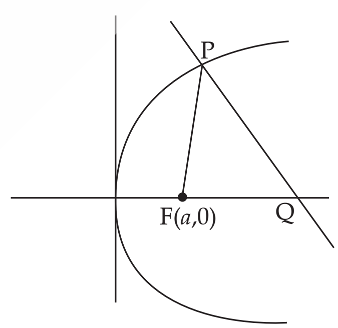
$$n(s) = {}^{12}C_3$$

A is even of selecting 3 points for which area of Δ is positive integer.

$$n(A) = 4 \times 7 + 9 \times 5 = 73$$

$$P(A) = \frac{73}{{}^{12}C_3} = \frac{73}{220}$$

7. Correct option is (A).



$$y^2 = 4ax$$

Equation of normal

$y = mx - 2am - am^3$

Point of contact

$P(am^2, -2am)$

and Point $Q\ (2a + am^2, 0)$

$$\text{Area of } \Delta\, PFQ = \frac{1}{2} \times |a + am^2|\ |-2am|$$

$$120 = a^2(1 + m^2)m \qquad \text{...(1)}$$

$$a = 2, m = 3$$

Satisfies the equation (1), hence (2, 3) will be the correct answer.

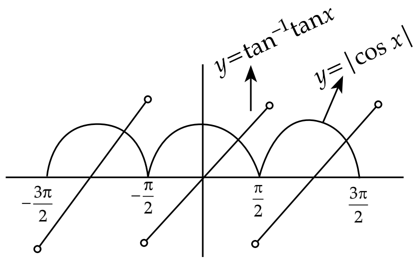
8. Correct answer is [3].

$$\sqrt{1 + \cos 2x} = \sqrt{2}\tan^{-1}(\tan x)$$

$\Rightarrow \sqrt{2\cos^2 x} = \sqrt{2}\ \tan^{-1}\tan x$

$\Rightarrow \sqrt{2}\ |\cos x| = \sqrt{2}\tan^{-1}\tan x$

$\Rightarrow \quad |\cos x| = \tan^{-1}\tan x$

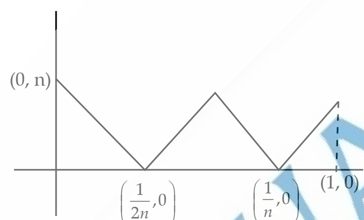


Number of solution = 3.

9. Correct option is [8].

$f : [0, 1] \rightarrow R$

$$f(x) = \begin{cases} n(1-2nx) & 0 \le x \le \frac{1}{2n} \\ 2n(2nx-1) & \frac{1}{2n} \le x \le \frac{3}{4n} \\ 4n(1-nx) & \frac{3}{4n} \le x \le \frac{1}{n} \\ \frac{n}{n-1}(nx-1) & \frac{1}{n} \le x \le 1 \end{cases}$$



$$\text{Area} = \frac{1}{2}\times\frac{1}{2n}\times n + \frac{1}{2}\times\frac{1}{2n}\times n + \frac{1}{2}\times\left(1-\frac{1}{n}\right)\times n$$

$$4 = \frac{1}{4}+\frac{1}{4}+\frac{n-1}{2}$$

$$4 = \frac{1}{2}+\frac{n-1}{2}$$

$$4 = \frac{n}{2}$$

$n = 8$

10. Correct answer is [1219].

$$S = 77 + 757 + 7557 + \ldots \overbrace{755\ldots57}^{98\text{ times}}$$

$$S = 70 + 700 + 7000 + \ldots \overbrace{70000\ldots00}^{99\text{ times}}$$

$$+ \underbrace{(50 + 550 + 5550 + \ldots.)}_{98\text{times}}$$

Let T_r be the general term.

$$T_r = 7\times10^{r-1} + 5(10 + 100 + \ldots 10^{r-2}) + 7\ r \ge 2$$

$$= 7\times10^{r-1} + 5\left[\frac{10(1-10^{r-2})}{1-10}\right] + 7$$

$$= 7\times10^{r-1} + \frac{50}{9}(10^{r-2}-1) + 7$$

$$= 7\times10^{r-1} + \frac{50}{9}(10^{r-2}) - \frac{50}{9} + 7$$

$$= 7\times10^{r-1} + \frac{50}{9}10^{r-2} + \frac{13}{9}$$

$$S = \sum_{r=2}^{100} T_r = \sum_{r=2}^{100} 7\times10^{r-1} + \frac{50}{90}\times10^{r-2} + \frac{13}{9}$$

$$= \frac{70}{9}(10^{99}-1) + \frac{50}{81}(10^{99}-1)\ \times 13\times 11$$

$$\text{RHS} = \frac{\overbrace{7555\ldots57}^{99\text{times}} + m}{n}$$

$$\frac{7\times10^{100} + \frac{50}{9}(10^{99}) + \frac{13}{9} + m}{n}$$

Now,

$$\frac{70}{9}(10^{99}-1) + \frac{50}{81}(10^{99}-1)\ 13\times 11$$

$$= \frac{\frac{70}{9}10^{100} + \frac{50}{9}\times10^{99} + \frac{13}{9} + m}{n}$$

$$= \frac{7}{n} + 10100 + \frac{50}{9n}1099 + \frac{13}{9n} + \frac{m}{n}$$

By Comparison,

$9 = n$ or $81 = 9n \Rightarrow n = 9$

$\therefore$ Put $n = 9$

$13\times 11\times 9^2 - 50 = 13 + 9m$

$m = 1210$

$\therefore \quad m + n = 1219$

11. Correct answer is [281].

$$A = \left\{\frac{1967 + 1686i\sin\theta}{7-3i\cos\theta}, \theta \in R\right\}$$

$\because$ A contains exactly one positive integer n.

Now simplifying

$$Z = \frac{1967 + 4686i\cos\theta}{7-3i\cos\theta}$$

$$= 281\frac{(7+6i\sin\theta)}{7-3i\cos\theta}\times\frac{7+3i\cos\theta}{7+3i\cos\theta}$$

$$= 281\frac{(49-9\sin 2\theta)}{49+9\cos^2\theta} + \frac{281\ (3)(2\sin\theta+\cos\theta)}{49+9\cos^2\theta}i$$

$$= 281\left(\frac{49-9\sin 2\theta}{49+9\cos^2\theta}\right) + 562\left(\frac{2\sin\theta+\cos\theta}{49+9\cos^2\theta}\right)i$$

For positive integer $Im(z) = 0$

We get, $2 \sin\theta + \cos\theta = 0$

$$\tan\theta = \frac{-1}{2}$$

$$\Rightarrow \quad \cos^2\theta = \frac{4}{5}$$

$$\Rightarrow \quad \sin 2\theta = \frac{2 + \tan\theta}{1 + \tan^2\theta}$$

$$= \frac{-1}{1 + \frac{1}{4}} = \frac{-4}{5}$$

$$\therefore \quad Z = 281\frac{\left(49 - 9\left(\frac{-4}{5}\right)\right)}{49 + 9\left(\frac{4}{5}\right)}$$

$$= 281$$

$$\therefore \quad n = 281$$

12. Correct answer is [45].

$P: \sqrt{3}x + 2y + 3z = 16$

$$S = \left\{\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k} : \alpha^2 + \beta^2 + \gamma^2 = 1, \quad d_p = \frac{7}{2}\right\}$$

$\because |\vec{u} - \vec{v}| = |\vec{v} - \vec{w}| = |\vec{w} - \vec{u}|$ ...(1)

$\vec{u}, \vec{v}, \vec{w}$ are elements of set S and in set S magnitude of vector is 1

$\therefore$ $\vec{u}, \vec{v}, \vec{w}$ are unit vectors and by equation (1) we can system $\vec{u}, \vec{v}, \vec{w}$ are equally inclined and vertices of equilateral triangle also lying on a circle which is intersection of sphere $|\vec{r}| = 1$

Distance from origin to P,

$$d = \frac{|-16|}{\sqrt{3 + 4 + 9}} = \frac{16}{4} = 4$$

$\therefore$ Plane containing $\hat{u}, \hat{v}, \hat{w}$ are at a distance $4 - \frac{7}{2} = \frac{1}{2}$ from origin and Parallel to $\sqrt{3}x + 2y + 3z = 16$.

$\therefore$ Equation of the plane is

$$\sqrt{3}x + 2y + 3z = \gamma$$

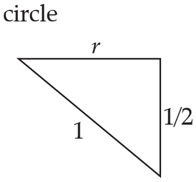
$$\therefore \quad \frac{1}{2} = \left|\frac{\gamma}{4}\right|$$

$$\Rightarrow \quad \gamma = \pm 2$$

$$\sqrt{3}x + 2y + 3z = 2$$

Equation of sphere $x^2 + y^2 + z^2 = 1$

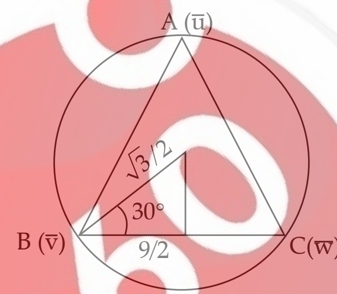
$\therefore$ Radius or circle



$$r = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$$

then $\quad \frac{a}{2} = \frac{\sqrt{3}}{2}\cos 30°$

$$a = \sqrt{3} \times \frac{\sqrt{3}}{2} = \frac{3}{2}$$



$\therefore$ Area or triangle

$$= \frac{\sqrt{3}}{2}a^2 = \frac{\sqrt{3}}{2} \times \frac{9}{4} = \frac{9\sqrt{3}}{16}$$

$\therefore$ Velocity of Parallelepiped

$$= 2 \times \frac{1}{2} \times \frac{9\sqrt{3}}{16}$$

$$V = \frac{9\sqrt{3}}{16}$$

$$\therefore \quad \frac{80V}{\sqrt{3}} = \frac{80}{\sqrt{3}} \times \frac{9\sqrt{3}}{16} = 45$$

13. Correct answer is [3].

General term of $\left(ax^2 + \frac{70}{27bx}\right)^4$

$$T_{r+1} = {}^4C_r\,(ax^2)^{4-r}\left(\frac{70}{27bx}\right)^r$$

$$= {}^4C_r\,a^{4-r}\,\frac{70^r}{(27b)^r}\,(x^{8-3r})$$

For Coefficient of x^5

$$8 - 3r = 5$$

$$r = 1$$

$$\therefore \quad \text{Coefficient} = {}^4C_1\,a^3 \cdot \frac{70}{27b}$$

$$= \frac{280}{27}\frac{a^3}{2}$$

General term of $\left(ax - \frac{1}{bx^2}\right)^7$ is

$$T_{r+1} = {}^7C_r\,(ax)^{7-r}\left(\frac{-1}{bx^2}\right)^r$$

$$= {}^7C_r\,a^{7-r}\left(-\frac{1}{b}\right)^r x^{7-3r}$$

For Coefficient of x^{-5}

$$7 - 3r = -5$$
$$r = 4$$

$\therefore$ Coefficient $= {}^7C_4\, a^3 \times \dfrac{1}{b^4}$

$\therefore$ According to the question,

$$\frac{280}{27}\frac{a^3}{b} = \frac{35 \times a^3}{b^4}$$

$$\Rightarrow \quad b^3 = \frac{27}{8}$$

$$\Rightarrow \quad b = \frac{3}{2}$$

$$\therefore \quad 2b = 3$$

14. Correct option is (A).

Given $x + 2y + z = 7$

$$x + \alpha z = 11$$
$$2x - 3y + \beta z = \gamma$$

Using Cramer's rule

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & \alpha \\ 2 & -3 & \beta \end{vmatrix}$$

$= 1(3\alpha) - 2(\beta - 2\alpha) + 1(-3)$

$= 3\alpha - 2\beta + 4\alpha - 3$

$= 7\alpha - 2\beta - 3$

$$\Delta_x = \begin{vmatrix} 7 & 2 & 1 \\ 11 & 0 & \alpha \\ \gamma & -3 & \beta \end{vmatrix}$$

$= 7(3\alpha) - 2(11\beta - \gamma\alpha) + 1(-33)$

$= 21\alpha - 22\beta + 22\gamma - 33$

$$\Delta y = \begin{vmatrix} 1 & 7 & 1 \\ 1 & 11 & \alpha \\ 2 & \gamma & \beta \end{vmatrix}$$

$= 1(11\beta - \alpha\gamma) - 7(\beta - 2\alpha) + 1(\gamma - 22)$

$= 11\beta - \alpha\gamma - 7\beta + 14\alpha + \gamma - 22$

$= 14\alpha + 4\beta + \gamma - \alpha\gamma - 22$

$$\Delta z = \begin{vmatrix} 1 & 2 & 7 \\ 1 & 0 & 11 \\ 2 & -3 & \gamma \end{vmatrix}$$

$= 1(33) - 2(\gamma - 22) + 7(-3)$

$= 33 - 2\gamma + 44 - 21$

$= -2\gamma + 56$

For unique solution $\Delta \neq 0$

For infinite solution

$$\Delta = \Delta x = \Delta y = \Delta z = 0$$

For no solution $\Delta = 0$ and atleast one in $\Delta x, \Delta y, \Delta z$ is non zero.

$$\Delta = 0$$

$$\Rightarrow \quad \beta = \frac{1}{2}(7\alpha - 3)$$

(P) $\quad \beta = \dfrac{1}{2}(7\alpha - 3)$ and $\gamma = 28$

then $\quad \Delta = 0, \Delta x = \Delta y = \Delta z = 0$

$\therefore$ Infinite solution

(Q) $\quad \beta = \dfrac{1}{2}(7\alpha - 3)$ and $\gamma \neq 28$

$\therefore \quad \Delta = 0$ and $\Delta_2 \neq 0$

$\Rightarrow$ No solution

(R) $\beta \neq \dfrac{1}{2}(7\alpha - 3)$, $\alpha = 1, \gamma \neq 28$

$\Rightarrow \quad \Delta \neq 0 =$ unique solution

(S) $\quad \beta \neq \dfrac{1}{2}(7\alpha - 3)$, $\alpha = 1, \gamma = 28$

$\therefore \quad \Delta \neq 0, \quad \Delta = 4 - 2\beta$

$\Delta x = 44 - 22\beta$

$\Delta y = 4\beta - 8$

$\Delta z = 0$

$\therefore \quad x = 11, y = -2, z = 0$ is the solution.

15. Correct option is (A).

x_i	f_i	$f_i x_i$	$f_i\lvert x_i - \bar{x}\rvert$	$f_i\lvert x_i - N\rvert$
3	5	15	15	10
4	4	16	8	4
5	4	20	4	0
8	2	16	4	6
10	2	20	8	10
11	3	33	15	18
	$\Sigma f_i = 20$	$\Sigma f_i x_i = 120$	sum = 54	sum = 48

(P) $\quad$ Mean $= \dfrac{\Sigma f_i x_i}{\Sigma f_i} = \dfrac{120}{20} = 6$

(Q) $\quad$ Median $= \dfrac{(10^{th} + 11^{th})\text{observation}}{2}$

$= \dfrac{5 + 5}{2} = 5$

(both observation are same)

(R) $\quad$ Mean deviation

$= \dfrac{\Sigma f_i \mid x_i - \bar{x} \mid}{\Sigma f_i} = \dfrac{54}{20}$

$= 2.7$

(S) $\quad$ Mean deviation about median

$= \dfrac{\Sigma f_i \mid x_i - M \mid}{\Sigma f_i} = \dfrac{48}{20}$

$= 2.4$

16. Correct option is (B).

$l_1: \quad \vec{r} = \lambda(\hat{i} + \hat{j} + \hat{k})$

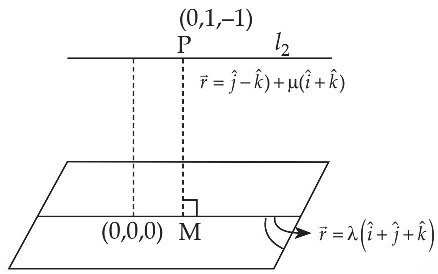
$l_2: \quad \vec{r} = \hat{j} - \hat{k} + \mu(\hat{i} + \hat{k})$

For plane

$d(H)$ = Smallest possible distance between the points of l_2 and Plane.

$d(H_0)$ = Maximum value of $d(H)$

For $d(H_0)$



l_2 is Parallel to plane containing l_1

Equation of plane

$a(x) + by + cz = 0$

$a(x) + by + cz = 0$ → $\vec{n} \perp l_1$, $\vec{n} \perp l_2$

$\therefore \quad a + b + c = 0$ ...(1)

$a + c = 0$ ...(2)

By (1) and (2) $a = -c, b = 0$

$\therefore$ Equation of plane $x - z = 0$

(P) $d(H_0) = PM = \left|\dfrac{0-(-1)}{\sqrt{1+1}}\right| = \dfrac{1}{\sqrt{2}}$

(Q) Distance from (0, 1, 2)

$= \left|\dfrac{0-2}{\sqrt{2}}\right| = \sqrt{2}$

(R) Distance from origin (0, 0, 0)

$= \left|\dfrac{0}{\sqrt{2}}\right| = 0$

(S) Point of Intersection,

$x - z = 0$ ...(1)

and $x = 1, y = z$ ...(2)

$\therefore \quad x = 1 = z = y$

$\therefore$ Point of intersection (1, 1, 1)

Distance from origin

$= \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

17. Correct option is (B).

$|Z|^3 + 2Z^2 + 4\bar{Z} - 8 = 0$

let $Z = x + iy$

$|Z| = \sqrt{x^2 + y^2}$

$\bar{Z} = x - iy$

$Z^2 = x^2 - y^2 + 2ixy$

$\therefore \quad |Z|^3 + 2Z^2 + 4\bar{Z} - 8 = 0$

$(x^2 + y^2)^{3/2} + 2(x^2 - y^2) + 4ixy + 4x - 4iy - 8 = 0$

$\therefore \quad (x^2 + y^2)^{3/2} + 2(x^2 - y^2) + 4x - 8 = 0$ ...(1)

and $2xy - 4y = 0$

$\Rightarrow \quad y = 0$ or $x = 1$

At $x = 1$

$(1 + y^2)^{3/2} + 2 - 2y^2 + 4 - 8 = 0$

$\Rightarrow (1 + y^2)^{3/2} - 2y^2 - 2 = 0$

$\Rightarrow (1 + y^2)^{3/2} - 2(1 + y^2) = 0$

$(1 + y^2)\left(\sqrt{1 + y^2} - 2\right) = 0$

then $1 + y^2 = 0$ (wich is not possible)

or $1 + y^2 = 4$

$\Rightarrow \quad y^2 = 3$

$\therefore \quad x = 1$ and $y^2 = 3$

(P) $|Z|^2 = x^2 + y^2 = 1 + 3 = 4$

(Q) $|Z - \bar{Z}|^2 = |2Im(z)|^2$

$= (2y)^2 = 4y^2 = 12$

(R) $|Z|^2 + |Z + \bar{Z}|^2 = 4 + |2x|^2$

$= 4 + 4(1) = 8$

(S) $|Z + 1|^2 = |x + iy + 1|^2$

$= (x + 1)^2 + y^2$

$= 4 + 3 = 7.$