JEE (Main) MATHEMATICS SOLVED PAPER

2021
16th March Shift 1

Time : 1 Hour **Total Marks : 100**

General Instructions :

1. *In Chemistry Section, there are 30 Questions (Q. no. 1 to 30).*
2. *In Chemistry, Section A consists of 20 multiple choice questions & Section B consists of 10 numerical value type questions. In Section B, candidates have to attempt any five questions out of 10.*
3. *There will be only one correct choice in the given four choices in Section A. For each question for Section A, 4 marks will be awarded for correct choice, 1 mark will be deducted for incorrect choice questions and zero mark will be awarded for not attempted question.*
4. *For Section B questions, 4 marks will be awarded for correct answer and zero for unattempted and incorrect answer.*
5. *Any textual, printed or written material, mobile phones, calculator etc. is not allowed for the students appearing for the test.*
6. *All calculations / written work should be done in the rough sheet is provided with Question Paper.*

Mathematics

Section A

Q. 1. Consider three observations a, b and c such that $b = a + c$. If the standard deviation of $a + 2$, $b + 2$, $c + 2$ is d, then which of the following is true?

(1) $b^2 = a^2 + c^2 + 3d^2$
(2) $b^2 = 3(a^2 + c^2) - 9d^2$
(3) $b^2 = 3(a^2 + c^2) + 9d^2$
(4) $b^2 = 3(a^2 + c^2 + d^2)$

Q. 2. Let a vector $\alpha\hat{i}+\beta\hat{j}$ be obtained by rotating the vector $\sqrt{3}\hat{i}+\hat{j}$ by an angle 45° about the origin in counter clockwise direction in the first quadrant. Then the area of triangle having vertices (α, β), $(0, \beta)$ and $(0, 0)$ is equal to :

(1) 1 **(2)** $\frac{1}{2}$
(3) $\frac{1}{\sqrt{2}}$ **(4)** $2\sqrt{2}$

Q. 3. If for $a > 0$, the feet of perpendiculars from the points A $(a, -2a, 3)$ and B $(0, 4, 5)$ on the plane $lx + my + nz = 0$ are points C $(0, -a, -1)$ and D respectively, then the length of line segment CD is equal to :

(1) $\sqrt{41}$ **(2)** $\sqrt{55}$
(3) $\sqrt{31}$ **(4)** $\sqrt{66}$

Q. 4. The range of $a \in$ R for which the function

$f(x) = (4a - 3)(x + \log_e 5)$

$+\ 2(a - 7)\cot\left(\frac{x}{2}\right)\sin^2\left(\frac{x}{2}\right), x \neq 2n\pi, n \in$ N

has critical points, is :

(1) $\left[-\frac{4}{3}, 2\right]$ **(2)** $[1, \infty)$
(3) $(-\infty, -1]$ **(4)** $(-3, 1)$

Q. 5. Let the functions f : R $\rightarrow$ R and g : R $\rightarrow$ R be defined as :

$f(x) = \begin{bmatrix} x+2, & x<0 \\ x^2, & x \geq 0 \end{bmatrix}$ and

$g(x) = \begin{cases} x^3, & x<1 \\ 3x-2, & x \geq 1 \end{cases}$

Then, the number of points in R where $(fog)(x)$ is NOT differentiable is equal to :

(1) 1 **(2)** 2
(3) 3 **(4)** 0

Q. 6. Let a complex number z, $|z| \neq 1$, satisfy $\log_{\frac{1}{\sqrt{2}}}\left(\frac{|z|+11}{(|z|-1)^2}\right) \leq 2$. Then, the largest value of $|z|$ is equal to ____

(1) 5 **(2)** 8
(3) 6 **(4)** 7

Q. 7. A pack of cards has one card missing. Two cards are drawn randomly and are found to be spades. The probability that the missing card is not a spade, is :

(1) $\frac{3}{4}$ **(2)** $\frac{52}{867}$

(3) $\frac{39}{50}$ **(4)** $\frac{22}{425}$

Q. 8. If n is the number of irrational terms in the expansion of $\left(3^{\frac{1}{4}}+5^{\frac{1}{8}}\right)^{60}$, then $(n - 1)$ is divisible by :

(1) 8 **(2)** 26

(3) 7 **(4)** 30

Q. 9. Let the position vectors of two points P and Q be $3\hat{i}-\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}-4\hat{k}$, respectively. Let R and S be two points such that the direction ratios of lines PR and QS are $(4, -1, 2)$ and $(-2, 1, -2)$ respectively. Let lines PR and QS intersect at T. If the vector $\overrightarrow{TA}$ is perpendicular to both $\overline{PR}$ and $\overline{QS}$ and the length of vector $\overrightarrow{TA}$ is $\sqrt{5}$ units, then the modulus of a position vector of A is :

(1) $\sqrt{5}$ **(2)** $\sqrt{171}$

(3) $\sqrt{227}$ **(4)** $\sqrt{482}$

Q. 10. If the three normals drawn to the parabola, $y^2 = 2x$ pass through the point $(a, 0)$ $a \neq 0$, then 'a' must be greater than:

(1) 1 **(2)** $\frac{1}{2}$

(3) $-\frac{1}{2}$ **(4)** -1

Q. 11. Let $S_k = \sum_{r=1}^{k} \tan^{-1}\left(\frac{6^r}{2^{2r+1}+3^{2r+1}}\right)$. Then $\lim_{k\to\infty} S_k$ is equal to :

(1) $\tan^{-1}\left(\frac{3}{2}\right)$ **(2)** $\cot^{-1}\left(\frac{3}{2}\right)$

(3) $\frac{\pi}{2}$ **(4)** $\tan^{-1}(3)$

Q. 12. The number of roots of the equation, $(81)^{\sin^2 x}+(81)^{\cos^2 x}=30$ in the interval $[0, \pi]$ is equal to :

(1) 3 **(2)** 2

(3) 4 **(4)** 8

Q. 13. If $y = y(x)$ is the solution of the differential equation, $\frac{dy}{dx}+2y\tan x=\sin x, y\left(\frac{\pi}{3}\right)=0$, then the maximum value of the function $y(x)$ over R is equal to :

(1) 8 **(2)** $\frac{1}{2}$

(3) $-\frac{15}{4}$ **(4)** $\frac{1}{8}$

Q. 14. Which of the following Boolean expression is a tautology?

(1) $(p \wedge q) \wedge (p \to q)$

(2) $(p \wedge q) \vee (p \vee q)$

(3) $(p \wedge q) \vee (p \to q)$

(4) $(p \wedge q) \to (p \to q)$

Q. 15. Let $A=\begin{bmatrix} i & -i \\ -i & i \end{bmatrix}, i=\sqrt{-1}$. Then, the system of linear equations $A^8\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 8 \\ 64 \end{bmatrix}$ has :

(1) No solution

(2) Exactly two solutions

(3) A unique solution

(4) Infinitely many solutions

Q. 16. If for $x \in \left(0, \frac{\pi}{2}\right)$, $\log_{10}\sin x + \log_{10}\cos x = -1$ and $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$, $n > 0$, then the value of n is equal to :

(1) 16 **(2)** 20

(3) 12 **(4)** 9

Q. 17. The locus of the midpoints of the chord of the circle, $x^2 + y^2 = 25$ which is tangent to the hyperbola, $\frac{x^2}{9}-\frac{y^2}{16}=1$ is :

(1) $(x^2+y^2)^2-16x^2+9y^2=0$

(2) $(x^2+y^2)^2-9x^2+144y^2=0$

(3) $(x^2+y^2)^2-9x^2-16y^2=0$

(4) $(x^2+y^2)^2-9x^2+16y^2=0$

Q. 18. Let $[x]$ denote greatest integer less than or equal to x. If for $n \in N, (1-x+x^3)^n=\sum_{j=0}^{3n} a_j x^j$, then $\sum_{j=0}^{\left[\frac{3n}{2}\right]} a_{2j} + 4\sum_{j=0}^{\left[\frac{3n-1}{2}\right]} a_{2j+1}$ is equal to :

(1) 1 **(2)** n
(3) 2^{n-1} **(4)** 2

Q. 19. Let P be a plane $lx + my + nz = 0$ containing the line, $\frac{1-x}{1}=\frac{y+4}{2}=\frac{z+2}{3}$. If plane P divides the line segment AB joining points A(–3, –6, 1) and B(2, 4, –3) in ratio $k : 1$ then the value of k is equal to :

(1) 1.5 **(2)** 2
(3) 4 **(4)** 3

Q. 20. The number of elements in the set $\{x \in R : (|x| -3)\ |x + 4| = 6\}$ is equal to :

(1) 2 **(2)** 1
(3) 3 **(4)** 4

Section B

Q. 21. Let $f : (0, 2) \to R$ be defined as $f(x) = \log_2\left(1+\tan\left(\frac{\pi x}{4}\right)\right)$. Then,

$\lim_{n\to\infty}\frac{2}{n}\left(f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+...+f(1)\right)$ is equal to ______

Q. 22. The total number of 3 × 3 matrices A having entries from the set {0, 1, 2, 3} such that the sum of all the diagonal entries of AA^T is 9, is equal to ____

Q. 23. Let $f : R \to R$ be a continuous function such that $f(x) + f(x + 1) = 2$, for all $x \in R$. If $I_1=\int_0^8 f(x)dx$ and $I_2=\int_{-1}^{3} f(x)dx$, then the value of $I_1 + 2I_2$ is equal to ____

Q. 24. Consider an arithmetic series and a geometric series having four initial terms from the set {11, 8, 21, 16, 26, 32, 4}. If the last terms of these series are the maximum possible four digit numbers, then the number of common terms in these two series is equal to ______.

Q. 25. If the normal to the curve $y(x)=\int_0^x\left(2t^2-15t+10\right)dt$ at a point (a, b) is parallel to the line $x + 3y = -5$, $a > 1$, then the value of $|a + 6b|$ is equal to ______.

Q. 26. If $\lim_{x\to 0}\frac{ae^x-b\cos x+ce^{-x}}{x\sin x}=2$, then $a + b + c$ is equal to ______.

Q. 27. Let ABCD be a square of side of unit length. Let a circle C_1 centered at A with unit radius is drawn. Another circle C_2 which touches C_1 and the lines AD and AB are tangent to it, is also drawn. Let a tangent line from the point C to the circle C_2 meet the side AB at E. If the length of EB is $\alpha+\sqrt{3}\beta$, where α, β are integers, then $\alpha + \beta$ is equal to ______.

Q. 28. Let z and ω be two complex numbers such that $\omega = z\bar{z} - 2z + 2, \left|\frac{z+i}{z-3i}\right| = 1$ and Re (ω) has minimum value. Then, the minimum value of $n \in N$ for which ω^n is real, is equal to ____.

Q. 29. Let $P=\begin{bmatrix}-30 & 20 & 56\\ 90 & 140 & 112\\ 120 & 60 & 14\end{bmatrix}$

and $A=\begin{bmatrix}2 & 7 & \omega^2\\ -1 & -\omega & 1\\ 0 & -\omega & -\omega+1\end{bmatrix}$

where $\omega=\frac{-1+i\sqrt{3}}{2}$, and I_3 be the identity matrix of order 3. If the determinant of the matrix $\left(P^{-1}AP-I_3\right)^2$ is $\alpha\omega^2$, then the value of α is equal to _______.

Q. 30. Let the curve $y = y(x)$ be the solution of the differential equation, $\frac{dy}{dx}=2(x+1)$. If the numerical value of area bounded by the curve $y = y(x)$ and x–axis is $\frac{4\sqrt{8}}{3}$, then the value of $y(1)$ is equal to______.

Answer Key

Q. No.	Answer	Topic Name	Chapter Name
1	2	Standard Deviation	Statistics
2	2	Application of Vector	Vector
3	4	Position of A Point With Respect To A Given Plane	Three Dimensional Geometry
4	1	Critical Point and Range of the Function	Relations and Functions
5	1	Derivability of Composite Function	Differentiability
6	4	Modules of Complex Number and Logarithmic Inequality	Complex Number
7	3	Conditional Probability	Probability
8	2	Rational / Irrational Terms	Binomial Theorem
9	2	Operation On Vectors	Vector
10	1	Normal of Parabola	Parabola
11	2	Summation of Series and Basic of Limits	Inverse Trigonometric Function
12	3	Methods of Solving Trigonometric Equation	Trigonometric Equation
13	4	Linear Differential Equation	Differential Equation
14	4	Tautology	Reasoning
15	1	System of Linear Equation	Matrices
16	3	Methods of Solving Trigonometric Equation, Logarithm	Trigonometric Equation
17	4	Tangent to the Hyperbola	Hyperbola
18	1	Sum of Binomial Coefficients	Binomial Theorem
19	2	Line and Plane	Three Dimensional Geometry
20	1	Application of Modules Function	Function
21	1	Limits Using Definite Integration	Definite Integration
22	766	Summation of Number, Permutation of Alike Objects of One Kind and Some Another Kinds, Transpose of Matrix, Product of Two Matrix	Permutation and Combination , Matrix
23	16	Properties of Periodic Function	Definite Integration
24	3	Arithmetic and Geometric Series	Progression
25	406	Derivatives of Anti Derivatives (Leibniz's Rule)	Definite Integration
26	4	Method To Solve Limits	Limits
27	1	Circle and Related Important Terms, Tangent To A Circle	Circle
28	4	Properties of Conjugate, Modulus and Argument	Complex Number
29.	36	Application of Matrix and Determinant	Matrix and Determinant
30	2	Area Between The Curves	Area

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ANSWERS WITH EXPLANATIONS

Mathematics

Section A

1. Option (2) is correct.

For three observations a, b and c,

$$\text{mean } (\bar{x}) = \frac{a+b+c}{3}$$

Also given that $b = a + c$

$$(\bar{x}) = \frac{2b}{3} \quad \text{...(i)}$$

Also given that standard deviation of $(a + 2)$, $(b + 2)$ and $(c + 2)$ is d.

S.D. $(a + 2, b + 2, c + 2) = d$

S.D $(a, b, c) = d$

$$d^2 = \frac{a^2+b^2+c^2}{3} - (\bar{x})^2 \text{; S.D} = \sqrt{\frac{\Sigma x^2}{n} - (\bar{x})^2}$$

$$d^2 = \frac{a^2+b^2+c^2}{3} - \left(\frac{2b}{3}\right)^2$$

$$d^2 = \frac{3(a^2+b^2+c^2) - 4b^2}{9}$$

$9d^2 = 3(a^2 + b^2 + c^2) - 4b^2$

$9d^2 = 3(a^2 + c^2) + 3b^2 - 4b^2$

$9d^2 = 3(a^2 + c^2) - b^2$

$b^2 = 3(a^2 + c^2) - 9d^2$

Hint:

(i) Mean of $[x_1, x_3, ...x_n]$

$$= \frac{x_1 + x_2 + ... + x_n}{n}$$

(ii) $\text{S.D} = \sqrt{\frac{\Sigma x^2}{n} - (\bar{x})^2}$

Shortcut Method:

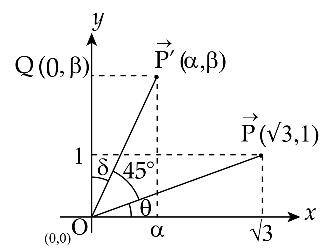
$$\bar{x} = \frac{a+b+c}{3} = \frac{2b}{3}$$

S.D $\{a, b, c\} = d$

$$d^2 = \frac{a^2+b^2+c^2}{3} - \frac{4b^2}{9}$$

$$\Rightarrow b^2 = 3(a^2+c^2) - 9d^2$$

2. Option (2) is correct.



Let $\vec{P} = \sqrt{3}\hat{i} + \hat{j} \Rightarrow \tan\theta = \left(\frac{1}{\sqrt{3}}\right) \Rightarrow \theta = 30°$

$\vec{P}' = \alpha\hat{i} + \beta\hat{j}$

$\theta + 45° + \delta = 90° \Rightarrow \delta = 45° - 30°$

$\delta = 15°$

Now, Area of $\Delta OP'Q$

$= \frac{1}{2}(OP' \cos 15°) \times (OP' \sin 15°)$

$= \frac{1}{4}(2 \sin 15° \cos 15°)(OP')^2$

$= \frac{1}{4}(\sin 30°)(OP')^2$

$= \frac{1}{4}\left(\frac{1}{2}\right)\left(\sqrt{(3)^2 + 1^2}\right) = \frac{1}{2}$ $\quad [\because OP = OP']$

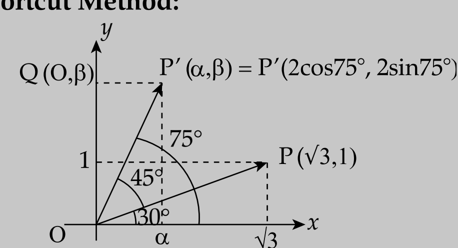
Hint:

(i) If $\vec{A} = a\hat{i} + b\hat{j}$ then $\tan\theta = \left(\frac{b}{a}\right)$

(ii) Length of segment AB will remain same after rotating in any direction.

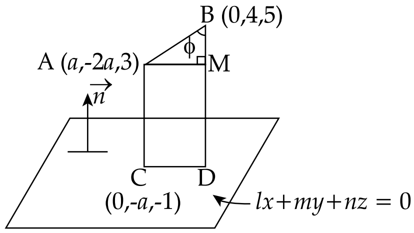
(iii) $\sin 2A = 2\sin A \cos A$

Shortcut Method:



$$\text{Area}(OP'Q) = \frac{1}{2}(2\cos 75°)(2\sin 75°) = \frac{1}{2}$$

3. Option (4) is correct.



$$CD = AM \quad ...(i)$$

In ΔABM: $\sin\phi = \frac{AM}{|AB|}$ ...(ii)

Using equation (i) and (ii)

$$AM = CD = |AB| \sin\phi$$

$$CD = |AB|\left(\sqrt{1-\cos^2\phi}\right)$$

$$\Rightarrow CD = |AB|\sqrt{1-\left(\frac{\overrightarrow{AB}\cdot\vec{n}}{|\overrightarrow{AB}|}\right)^2} \quad \left(\because \cos\phi = \frac{\overrightarrow{AB}\cdot\vec{n}}{|\vec{n}||\overrightarrow{AB}|}\right)$$

$$\Rightarrow CD = \sqrt{\left(|\overrightarrow{AB}|\right)^2 - \left(\overrightarrow{AB}\cdot\vec{n}\right)^2} \quad ...(iii)$$

As A $(a, -2a, 3)$ and B $(0, 4, 5)$ then

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -a\hat{i} + (2a+4)\hat{j} + 2\hat{k}$$

$$\overrightarrow{AB}\cdot\vec{n} = -la + (2a+4)m + 2n \quad ...(iv)$$

Since, C$(0, -a, -1)$ lies on plane $lx + my + nz = 0$,

So,

$$0l - am - n = 0 \Rightarrow \frac{m}{n} = \frac{-1}{a} \quad ...(v)$$

From the figure

$$\overrightarrow{AC} \parallel \vec{n}$$

$$\frac{a}{l} = \frac{-a}{m} = \frac{4}{n}$$

$$m = -l \text{ and } \frac{m}{n} = \frac{-a}{4} \quad ...(vi)$$

Now using equation (v) and (vi)

$\Rightarrow a^2 = 4$

$$a = \pm 2$$

$$\text{As } a > 0,\ a = 2$$

Now from equation (vii)

$$2m + n = 0 \quad ...(vii)$$

$$[\text{As } l^2 + m^2 + n^2 = 1]$$

$$\therefore\ m^2 + m^2 + 4m^2 = 1$$

$$\therefore\ m^2 = \frac{1}{6}$$

$$\therefore\ m = \pm\frac{1}{\sqrt{6}}$$

$$\therefore\ m = \frac{1}{\sqrt{6}}$$

Using equation (vii)

$$n = -2m$$

$$n = \frac{-2}{\sqrt{6}}$$

$$l = \frac{-1}{\sqrt{6}}$$

Now from equation (iv)

$$\overrightarrow{AB}\cdot\vec{n} = -2\left(\frac{-1}{\sqrt{6}}\right) + 8\left(\frac{1}{\sqrt{6}}\right) + 2\left(\frac{-2}{\sqrt{6}}\right)$$

$$= \frac{+2+8-4}{\sqrt{6}}$$

$$= \sqrt{6}$$

$$\overrightarrow{AB} = \sqrt{a^2 + (2a+4)^2 + (2)^2}$$

$$= \sqrt{2^2 + 8^2 + 4^2}$$

$$|\overrightarrow{AB}| = \sqrt{4+64+4} = \sqrt{72}$$

$$CD = \sqrt{\left(\sqrt{72}\right)^2 - \left(\sqrt{6}\right)^2}$$

$$CD = \sqrt{72-6}$$

$$CD = \sqrt{66}$$

Hint:

(i) For two vectors, $\vec{A}$ and $\vec{B}$

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|A||B|}$$

(ii) If $\vec{A} = a\hat{i} + b\hat{j} + c\hat{k}$ and

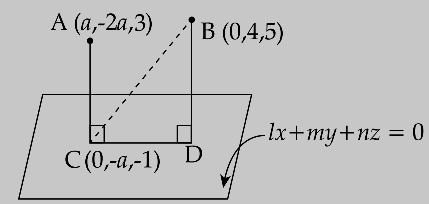
$\vec{B} = l\hat{i} + m\hat{j} + n\hat{k}$ and

$\vec{A} = \vec{B}$ then

$$\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$$

(iii) If a point lies on a plane then it always satisfy that plane.

Shortcut Method:



C lies on plane $lx + my + nz = 0$

$$\frac{m}{n} = -\frac{1}{a} \quad ...(i)$$

$$\overrightarrow{CA} \parallel l\hat{i} + m\hat{j} + n\hat{k}$$

$$\frac{a-o}{l} = \frac{-a}{m} = \frac{4}{n} \Rightarrow \frac{m}{n} = \frac{-a}{4} \quad ...(ii)$$

Using (i) and (ii)

$\Rightarrow \quad a = 2\ (a > 0)$

Using equation (ii)

$$\frac{m}{n} = \frac{-1}{2}$$

Let $\quad m = -\lambda$

$\Rightarrow \quad n = 2\lambda$

$$\frac{2}{l} = \frac{-2}{-\lambda}$$

$\Rightarrow \quad l = \lambda$

So, plane: $\lambda(x - y + 2z) = 0$

$BD = \sqrt{6}$; $C(0, -2, -1)$

$$CD = \sqrt{(BC)^2 - (BD)^2}$$

$$CD = \sqrt{66}$$

4. Option (1) is correct.

Given that

$f(x) = (4a - 3)(x + \log_e 5)$

$$+ 2(a - 7)\cot\left(\frac{x}{2}\right)\sin^2\left(\frac{x}{2}\right); x \neq 2n\pi, n \in N$$

$$f(x) = (4a - 3)(x + \log_e 5) + 2(a - 7)\left(\cos\frac{x}{2}\right)\left(\sin\frac{x}{2}\right)$$

$f(x) = (4a - 3)(x + \log_e 5) + (a - 7)(\sin x)$

Also given that $f(x)$ has critical points,

i.e., $f'(x) = 0$

$(4a - 3)(1 + 0) + (a - 7)(\cos x) = 0$

$$\cos x = \frac{3-4a}{a-7}$$

As, $\quad \cos x \in [-1, 1]$

$$\frac{3-4a}{a-7} \in [-1, 1]$$

$$-1 \leq \frac{3-4a}{a-7} \leq 1$$

$\frac{3-4a}{a-7} \geq -1$ and $\frac{3-4a}{a-7} \leq 1$

$\frac{3-4a}{a-7} + 1 \geq 0$ and $\frac{3-4a}{a-7} - 1 \leq 0$

$\frac{3-4a+a-7}{a-7} \geq 0$ and $\frac{3-4a-a+7}{a-7} \leq 0$

$\frac{-3a-4}{a-7} \geq 0$ and $\frac{10-5a}{a-7} \leq 0$

$\frac{a+\frac{4}{3}}{a-7} \leq 0$ and $\frac{a-2}{a-7} \geq 0$

$a \in \left[\frac{-4}{3}, 7\right)$ and $a \in (-\infty, 2] \cup (7, \infty)$

$$a \in \left[\frac{-4}{3}, 2\right]$$

Hint:

(i) $\sin 2A = 2\sin A \cos A$

(ii) $\cos x \in [-1, 1]$

(iii) For critical points, $\frac{dy}{dx} = 0$

Shortcut Method:

$f'(x) = 0$

$$\cos x = \frac{3-4a}{a-7}$$

$$-1 \leq \frac{3-4a}{a-7} \leq 1$$

$$a \in \left[\frac{-4}{3}, 2\right]$$

5. Option (1) is correct.

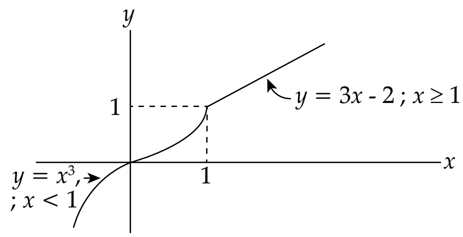
Given that

$$f(x) = \begin{cases} x+2 & ; \quad x < 0 \\ x^2 & ; \quad x \geq 0 \end{cases}; f: R \to R$$

$$g(x) = \begin{cases} x^3 & ; \quad x < 1 \\ 3x-2 & ; \quad x \geq 1 \end{cases}; g: R \to R$$

Now,

$$f(g(x)) = \begin{cases} g(x)+2 & ; \quad g(x) < 0 \\ (g(x))^2 & ; \quad g(x) \geq 0 \end{cases}$$



$$f(g(x)) = \begin{cases} x^3+2 & ; \quad x<0 \\ x^6 & ; \quad 0 \le x \le 1 \\ (3x-2)^2 & ; \quad x \ge 1 \end{cases}$$

For $f(g(x))$: $\lim\limits_{x\to 0^-}\left(x^3+2\right) \neq \lim\limits_{x\to 0^+}\left(x^6\right)$

Hence $f(g(x))$ is discontinuous at $x = 0$.
i.e. not differentiable at $x = 0$.

$$\lim_{x\to 1^-}\left(x^6\right) = \lim_{x\to 1^+}(3x-2)^2 = 1$$

i.e. $f(g(x))$ is continuous at $x = 1$

So, we will check differentiability at $x = 1$.

$$\text{R.H.D.}\Big|_{x=1} = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{(3(1+h)-2)^2-1}{h}$$

$$= \lim_{h\to 0}\frac{(3h+1)^2-1}{h}$$

$$= \lim_{h\to 0}\frac{9h^2+6h+1-1}{h}$$

$$= \lim_{h\to 0}(9h+6)$$

$$\text{R.H.D.}\Big|_{x=1} = 6$$

Now, $$\text{L.H.D.}\Big|_{x=1} = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{(1-h)^6-1}{-h}$$

$$\text{L.H.D.}\Big|_{x=1} = 6$$

$$\text{L.H.D.}\Big|_{x=1} = \text{R.H.D.}\Big|_{x=1} = 6$$

Hence $f(g(x))$ is differentiable everywhere except one point, *i.e.*, 0.
So, Number of points of non-differentiability = 1.

Hint:

(i) Using suitable method to find $f(g(x))$.

(ii) If a function is discontinuous at any point then it will be not differentiable at that point.

(iii) A function will be differentiable at $x = a$ if $\text{L.H.D}\big|_{x=a^-} = \text{R.H.D}\big|_{x=a^+}$ = Finite quantity

Shortcut Method:

$$fog\,(x) = \begin{cases} x^3+2 & ; \quad x<0 \\ x^6 & ; \quad 0 \le x \le 1 \\ (3x-2)^2 & ; \quad x \ge 1 \end{cases}$$

$$fog'\,(x) = \begin{cases} 3x^2 & ; \quad x<0 \\ 6x^5 & ; \quad 0 \le x \le 1 \\ 6(3x-2) & ; \quad x \ge 1 \end{cases}$$

At $x = 0$; LHD ≠ RHD

At $x = 1$; LHD = RHD

$fog(x)$ is not differentiable at $x = 0$

6. Option (4) is correct.

Given that $|z| \neq 1$

$$\log_{\frac{1}{\sqrt{2}}}\left(\frac{|z|+11}{(|z|-1)^2}\right) \le 2$$

Here base of logarithm lies between 0 and 1

So,

$$\Rightarrow \quad \frac{|z|+11}{(|z|-1)^2} \ge \left(\frac{1}{\sqrt{2}}\right)^2$$

$$\Rightarrow \quad \frac{|z|+11}{(|z|-1)^2} \ge \frac{1}{2}$$

$\Rightarrow \quad 2|z| + 22 \ge (|z| - 1)^2$

$\Rightarrow \quad 2|z| + 22 \ge |z|^2 - 2|z| + 1$

$\Rightarrow \quad |z|^2 - 4|z| - 21 \le 0$

$\Rightarrow \quad (|z| - 7)(|z| + 3) \le 0$

$\Rightarrow \quad |z| - 7 \le 0$

$\Rightarrow \quad |z| \le 7$

So, largest value of $|z|$ is 7.

Hint:

(i) $\log_a x \le m \Rightarrow x \ge a^m;\ a \in (0, 1)$

(ii) $|z| > 0$

Shortcut Method:

$$\frac{|z|+11}{(|z|-1)^2} \ge \frac{1}{2}$$

$|z| - 7 \le 0$

$|z| \le 7$

$|z|_{max} = 7$

7. Option (3) is correct.

Let, E_1 = Event in which spade is missing

$$P(E_1) = \frac{1}{4} \quad \text{...(i)}$$

$$P(\overline{E_1}) = 1 - P(E_1)$$

$$P(\overline{E_1}) = \frac{3}{4} \quad \text{...(ii)}$$

E = Event in which drawn two cards are spade.

$$P(E) = \frac{\left(\frac{1}{4}\right)\left(\frac{{}^{12}C_2}{{}^{51}C_2}\right)+\left(\frac{3}{4}\right)\left(\frac{{}^{13}C_2}{{}^{51}C_2}\right)-\left(\frac{3}{4}\right)\left(\frac{{}^{13}C_2}{{}^{51}C_2}\right)}{\left(\frac{1}{4}\right)\left(\frac{{}^{12}C_2}{{}^{51}C_2}\right)+\left(\frac{3}{4}\right)\left(\frac{{}^{13}C_2}{{}^{51}C_2}\right)}$$

$$P(E) = \frac{\left(\frac{1}{4}\right)\left(\frac{12\times11}{51\times50}\right)+\left(\frac{3}{4}\right)\left(\frac{13\times12}{51\times50}\right)-\left(\frac{3}{4}\right)\left(\frac{13\times12}{51\times50}\right)}{\left(\frac{1}{4}\right)\left(\frac{12\times11}{51\times50}\right)+\left(\frac{3}{4}\right)\left(\frac{13\times12}{51\times50}\right)}$$

$$P(E) = \frac{(12\times11)+(3)(13\times12)-(3)(13\times12)}{(12\times11)+(3)(13\times12)}$$

$$P(E) = \frac{11}{50}$$

Required probability = 1 – P(E)

$$= 1 - \frac{11}{50}$$

$$= \frac{39}{50}$$

Shortcut Method:

$P(\bar{S}_{missing}$ / Both found spade)

$$= \frac{p(\bar{S}_m \cap \text{Both found spade})}{p(\text{Both found spade})}$$

$$= \frac{\left(1-\frac{13}{52}\right)\left(\frac{13}{51}\right)\left(\frac{12}{50}\right)}{\left(1-\frac{13}{52}\right)\left(\frac{13}{51}\times\frac{12}{50}\right)+\left(\frac{13}{52}\times\frac{12}{51}\right)\times\frac{11}{50}}$$

$$= \frac{39}{50}$$

8. Option (2) is correct.

Given binomial expression is $\left(3^{\frac{1}{4}}+5^{\frac{1}{8}}\right)^{60}$

For $(A+B)^n$,

$$T_{r+1} = {}^nC_r(A)^{n-r}(B)^r;\ 0 \le r \le n;\ r \in W$$

Using above concept, we can write

$$T_{r+1} = {}^{60}C_r(3^{\frac{1}{4}})^{60-r}(5^{\frac{1}{8}})^r$$

$$\Rightarrow \quad T_{r+1} = {}^{60}C_r(3)^{\frac{60-r}{4}}(5)^{\frac{r}{8}} \quad \text{...(i)}$$

As $\quad 0 \le r \le n \Rightarrow 0 \le r \le 60 \quad$ ...(ii)

$$\Rightarrow \quad 0 \le \frac{r}{8} \le \frac{60}{8}$$

$$\Rightarrow \quad 0 \le \frac{r}{8} \le 7.5$$

For rational terms

$$\frac{r}{8} \in \{0, 1, 2, 3, 4, 5, 6, 7\} \quad \text{...(iii)}$$

Using equation (ii),

$0 \le r \le 60$

$-60 \le -r \le 0$

$0 \le 60 - r \le 60$

$0 \le \frac{60-r}{4} \le 15$

For rational terms

$$\frac{60-r}{4} \in \{0,1,2,3,4,.........,15\} \quad \text{...(iv)}$$

Using Equation (iii) and (iv),

Total rational terms = 8

Total number of terms = 60 + 1 = 61

Hence total irritational number of terms

= 61 – 8

= 53

Given, n = Number of irrational terms

$n = 53$

$\Rightarrow n-1 = 53-1$

$\Rightarrow n-1 = 52$

$\Rightarrow$ 52 is divisible by 26

Hint:

(i) For $(A+B)^n$

$T_{r+1} = {}^nC_r (A)^{n-r} (B)^r$

$0 \le r \le n$

$r \in W$

Number of dissimilar terms = $n + 1$.

(ii) Irrational number of terms = Total terms – Total rational terms

Shortcut Method:

$T_{r+1} = {}^{60}C_r (3)^{\frac{60-r}{4}} (5)^{\frac{r}{8}}$

$r \in \{0,1,2,.......60\}$

$\frac{r}{8} \in \{0,1,2,3,4,5,6,7\}$

$\frac{60-r}{4} \in \{0,1,2,.......,15\}$

Total number of irrational terms

$= 61 - 8 = 53$

$n = 53 \Rightarrow n - 1 = 52$, which is divisible by 26.

9. Option (2) is correct.

Given, $\vec{P} = 3\hat{i} - \hat{j} + 2\hat{k}$; P(3, – 1, 2)

and $\vec{Q} = \hat{i} + 2\hat{j} - 4\hat{k}$; Q(1, 2, – 4)

$\overrightarrow{PR} \parallel 4\hat{i} - \hat{j} + 2\hat{k}$ and $\overrightarrow{QS} \parallel -2\hat{i} + \hat{j} - 2\hat{k}$

Direction ratios of normal to the plane containing P, T and Q will proportional to

$$\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 2 \\ -2 & 1 & -2 \end{bmatrix}$$

$\Rightarrow 0\hat{i} + 4\hat{j} + 2\hat{k}$

$\therefore \quad \frac{l}{0} = \frac{m}{4} = \frac{n}{2}$

For the point, T:

$$\overrightarrow{PT} = \frac{x-3}{4} = \frac{y+1}{-1} = \frac{z-2}{2} = \lambda$$

and $\overrightarrow{QT} = \frac{x-1}{-2} = \frac{y-2}{1} = \frac{z+4}{-2} = \mu$

T : $(4\lambda+3, -\lambda-1, 2\lambda+2)$

$\simeq (-2\mu+1, \mu+2, -2\mu-4)$

After comparing, we get

$4\lambda + 3 = -2\mu + 1 \Rightarrow 2\lambda + \mu = -1$

$-\lambda - 1 = \mu + 2 \Rightarrow \lambda + \mu = -3$

$2\lambda + 2 = -2\mu - 4 \Rightarrow \lambda + \mu = -3$

$\Rightarrow \lambda = 2$ and $\mu = -5$

So, point T(11, –3, 6)

$\overrightarrow{OT} = 11\hat{i} - 3\hat{j} + 6\hat{k}$

Now, $\overrightarrow{TA} = \overrightarrow{OA} - \overrightarrow{OT}$

$\overrightarrow{OA} = \overrightarrow{OT} + \overrightarrow{TA}$

$$\overrightarrow{OA} = \left(11\hat{i} - 3\hat{j} + 6\hat{k}\right) + \frac{\left(4\hat{j} + 2\hat{k}\right)}{\sqrt{4^2 + 2^2}} \times \sqrt{5}$$

$$\overrightarrow{OA} = \left(11\hat{i} - 3\hat{j} + 6\hat{k}\right) + \frac{\left(4\hat{j} + 2\hat{k}\right)}{2\sqrt{5}} \times \sqrt{5}$$

$$\overrightarrow{OA} = (11\hat{i} - 3\hat{j} + 6\hat{k}) + \left(\frac{2\hat{j} + \hat{k}}{\sqrt{5}}\right)\left(\sqrt{5}\right)$$

$$\overrightarrow{OA} = (11\hat{i} - 3\hat{j} + 6\hat{k}) + (2\hat{j} + \hat{k})$$

$\overrightarrow{OA} = 11\hat{i} - \hat{j} + 7\hat{k} \Rightarrow \overrightarrow{OA} = \sqrt{171}$

Hint:

(i) If $\vec{A} = a\hat{i} + b\hat{j} + c\hat{k}$ and $\vec{B} = l\hat{i} + m\hat{k} + n\hat{j}$ are parallel vectors then

$\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$

(ii) $\vec{A} = a\hat{i} + b\hat{j} + c\hat{k} \Rightarrow$

$|\vec{A}| = \sqrt{a^2 + b^2 + c^2}$

10. Option (1) is correct.

Given,

parabola is $y^2 = 2x$ ...(i)

Let the equation of the normal is

$y = mx - 2am - am^3$ ...(ii)

Using equation (i)

$4a = 2$;

(Standard Equation of parabola $y^2 = 4ax$)

$\Rightarrow a = \frac{1}{2}$ ...(iii)

Using Equation (ii) and (iii)

$$y = mx - m - \frac{1}{2}m^3$$

Given that normal passes through the point $(a, 0)$

Hence,

$$0 = m(a) - m - \frac{1}{2}(m)^3$$

$$\Rightarrow m(a - 1 - \frac{m^2}{2}) = 0$$

$$\Rightarrow m = 0 \text{ or } a - 1 - \frac{m^2}{2} = 0$$

$$\Rightarrow m = 0 \text{ or } m^2 = 2(a-1)$$

As $m^2 > 0 \Rightarrow 2(a-1) > 0$

$a - 1 > 0$

$a > 1$

Hence, 'a' must be greater than one.

Hint:

(i) For the parabola $y^2 = 4ax$, Equation of normal , $y = mx - 2am - am^3$

(ii) If any point lies on the curve than that point will satisfy that curve.

(iii) If L.H.S > 0 then R.H.S > 0

Shortcut Method:

For standard parabola, $y^2 = 4ax$,

for more than 3 normals (on axis)

$x > \frac{L}{2}$; L = Length of latus rectum

Now, for $y^2 = 2x$; $\left(a = \frac{1}{2}\right)$

L.R = 2

for point $(a, 0)$

$a > \frac{L}{2} \Rightarrow a > 1.$

11. Option (2) is correct.

Given,

$$S_k = \sum_{r=1}^{k} \tan^{-1}\left(\frac{6^r}{2^{2r+1}+3^{2r+1}}\right)$$

$$\Rightarrow S_k = \sum_{r=1}^{k} \tan^{-1}\left(\frac{\frac{6^r}{9^r}}{\frac{2^{2r+1}+3^{2r+1}}{9^r}}\right)$$

$$\Rightarrow S_k = \sum_{r=1}^{k} \tan^{-1}\left\{\frac{\left(\frac{2^r.3^r}{3^r.3^r}\right)}{\left(\frac{2}{3}\right)^{2r}.2+\left(\frac{3^{2r}}{3^{2r}}\right).3}\right\}$$

$$\Rightarrow S_k = \sum_{r=1}^{k} \tan^{-1}\left\{\frac{\left(\frac{2}{3}\right)^r}{3+2.\left(\frac{2}{3}\right)^{2r}}\right\}$$

$$\Rightarrow S_k = \sum_{r=1}^{k} \tan^{-1}\left\{\frac{\left(\frac{2}{3}\right)^r}{3\left(1+\left(\frac{2}{3}\right).\left(\frac{2}{3}\right)^{2r}\right)}\right\}$$

Let $\left(\frac{2}{3}\right)^r = t$

$$\Rightarrow S_k = \sum_{r=1}^{k} \tan^{-1}\left\{\frac{t}{3.\left(1+\left(\frac{2}{3}\right)t^2\right)}\right\}$$

$$\Rightarrow S_k = \sum_{r=1}^{k} \tan^{-1}\left\{\frac{\frac{t}{3}}{1+(t).\left(\frac{2t}{3}\right)}\right\}$$

$$\Rightarrow S_k = \sum_{r=1}^{k} \tan^{-1}\left\{\frac{t-\frac{2t}{3}}{1+(t).\left(\frac{2t}{3}\right)}\right\}$$

$$\Rightarrow S_k = \sum_{r=1}^{k}\left\{\tan^{-1}(t) - \tan^{-1}\left(\frac{2t}{3}\right)\right\}$$

$$\Rightarrow S_k = \sum_{r=1}^{k}\left\{\tan^{-1}\left(\frac{2}{3}\right)^r - \tan^{-1}\left(\frac{2}{3}\right)^{r+1}\right\};$$

$$\Rightarrow S_k = \left\{\tan^{-1}\left(\frac{2}{3}\right) - \tan^{-1}\left(\frac{2}{3}\right)^2\right\} + \left\{\tan^{-1}\left(\frac{2}{3}\right)^2 - \tan^{-1}\left(\frac{2}{3}\right)^3\right\}$$

$$+\left\{\tan^{-1}\left(\frac{2}{3}\right)^3 - \tan^{-1}\left(\frac{2}{3}\right)^4\right\} + ----$$

$$---- + \left\{\tan^{-1}\left(\frac{2}{3}\right)^k - \tan^{-1}\left(\frac{2}{3}\right)^{k+1}\right\}$$

$$\Rightarrow S_k = \tan^{-1}\left(\frac{2}{3}\right) - \tan^{-1}\left(\frac{2}{3}\right)^{k+1}$$

$$\Rightarrow \lim_{k\to\infty}(S_k) = \lim_{k\to\infty}\left\{\tan^{-1}\left(\frac{2}{3}\right) - \tan^{-1}\left(\frac{2}{3}\right)^{k+1}\right\}$$

$$\Rightarrow S_\infty = \tan^{-1}\left(\frac{2}{3}\right) - \tan^{-1}(0);\ \lim_{k\to\infty}\left(\frac{2}{3}\right)^{k+1} = 0$$

$$\Rightarrow S_\infty = \tan^{-1}\left(\frac{2}{3}\right) - 0$$

$$\Rightarrow S_\infty = \tan^{-1}\left(\frac{2}{3}\right)$$

$$\Rightarrow S_\infty = \cot^{-1}\left(\frac{3}{2}\right)$$

Hint:

Divide numerator and denominator by 9^r. Then assume $\left(\frac{2}{3}\right)^r$ as t and simplify.

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

Use method of difference to simplify.

$$\lim_{k\to\infty}(x)^k = 0; 0 < x < 1$$

$$\tan^{-1}(x) = \cot^{-1}\left(\frac{1}{x}\right); 0 < x < 1$$

Shortcut Method:

$$\sum_{r=1}^{\infty}\tan^{-1}\left(\frac{6^r(3-2)}{\left(1+\left(\frac{3}{2}\right)^{2r+1}\right)2^{2r+1}}\right)$$

$$= \sum_{r=1}^{\infty}\tan^{-1}\left(\frac{\left(\frac{3}{2}\right)^{r+1} - \left(\frac{3}{2}\right)^{r}}{1+\left(\frac{3}{2}\right)^{r+1}\cdot\left(\frac{3}{2}\right)^{r}}\right)$$

$$= \sum_{r=1}^{\infty}\left\{\tan^{-1}\left(\frac{3}{2}\right)^{r+1} - \tan^{-1}\left(\frac{3}{2}\right)^{r}\right\}$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{3}{2}\right) = \cot^{-1}\left(\frac{3}{2}\right)$$

12. Option (3) is correct.

Given,

$$(81)^{\sin^2 x} + (81)^{\cos^2 x} = 30$$

$$(81)^{\sin^2 x} + (81)^{1-\sin^2 x} = 30$$

$$\Rightarrow (81)^{\sin^2 x} + (81)^1(81)^{-\sin^2 x} = 30$$

$$\Rightarrow (81)^{\sin^2 x} + \frac{81}{(81)^{\sin^2 x}} = 30$$

Let $(81)^{\sin^2 x} = t$

$$\Rightarrow t + \frac{81}{t} = 30$$

$$\Rightarrow t^2 - 30t + 81 = 0$$

$$\Rightarrow (t-27)(t-3) = 0$$

$$\Rightarrow t = 3 \text{ or } t = 27$$

$$\Rightarrow (81)^{\sin^2 x} = 3 \text{ or } (81)^{\sin^2 x} = 27$$

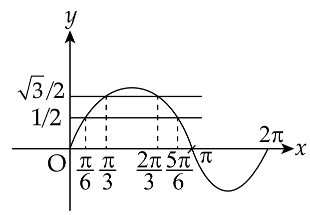
$$\Rightarrow (3^4)^{\sin^2 x} = 3 \text{ or } (3^4)^{\sin^2 x} = 3^3$$

$$\Rightarrow 3^{4\sin^2 x} = 3^1 \text{ or } 3^{4\sin^2 x} = 3^3$$

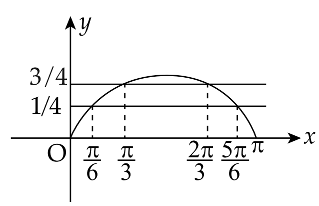
$$\Rightarrow 4\sin^2 x = 1 \text{ or } 4\sin^2 x = 3$$

$$\Rightarrow \sin^2 x = \frac{1}{4} \text{ or } \sin^2 x = \frac{3}{4}$$

Now $y = \sin x$



For $y = \sin^2 x; x \in [0, \pi]$



From the above figure, we can say that the given equation has 4 solution.

Hint:

(i) Replace $\cos^2 x = 1 - \sin^2 x$

(ii) Assume $(81)^{\sin^2 x} = t$ and simplify.

(iii) Draw the graph of $\sin^2 x$ and count number of intersection points for $\sin^2 x = C$, $C \in$ constant.

Shortcut Method:

$(81)^{\sin^2 x}+\frac{81}{(81)^{\sin^2 x}}=30\ ;\ x\in[0,\pi]$

$\Rightarrow\ (81)^{\sin^2 x}=3,27$

$\Rightarrow \sin^2 x=\frac{1}{4},\frac{3}{4}\Rightarrow x=\frac{\pi}{6},\frac{5\pi}{6},\frac{\pi}{3},\frac{2\pi}{3}$

So, number of solution = 4

13. Option (4) is correct.

Given differential equation is

$$\frac{dy}{dx}+2y\tan x=\sin x \quad ...(i)$$

Solution of this differential equation will be

$$y\text{ (I.F)} = \int \sin x(\text{I.F})dx \quad ...(ii)$$

$$\text{I.F} = e^{\int 2\tan x dx}$$

$$\text{I.F} = e^{2\int \tan x dx}$$

$$\because \int \tan x\ dx = \ln(\sec x)+c$$

$$\text{I.F.} = e^{2ln\sec x}$$

$$\text{I.F} = e^{\ln \sec^2 x}$$

$$\text{I.F} = \sec^2 x \quad ...(iii)$$

Now using equation (ii) and (iii)

$$y(\sec^2 x)=\int(\sin x)(\sec^2 x)dx$$

$$\Rightarrow y(\sec^2 x)=\int\left(\frac{\sin x}{\cos x}\right)(\sec x)dx$$

$$\Rightarrow y(\sec^2 x)=\int(\tan x\sec x)dx$$

$$\Rightarrow y\left(\sec^2 x\right)=\sec x+\text{C} \quad ...(iv)$$

Given that $y\left(\frac{\pi}{3}\right)=0$ *i.e.*, when $x=\frac{\pi}{3}, y=0$

$$\Rightarrow (0)\left(\sec^2\frac{\pi}{3}\right)=\sec\left(\frac{\pi}{3}\right)+\text{C}\Rightarrow \text{C}=-\sec\frac{\pi}{3}$$

$$\Rightarrow \text{C} = -2 \quad ...(v)$$

Using equation (iv) and (v)

$$y(\sec^2 x)=\sec x-2$$

$$\Rightarrow y=\frac{\sec x-2}{\sec^2 x}$$

$$\Rightarrow y=\frac{\sec x}{\sec^2 x}-\frac{2}{\sec^2 x}$$

$$\Rightarrow y=\frac{1}{\sec x}-\frac{2}{\sec^2 x}$$

$$\Rightarrow y=\cos x-2\cos^2 x$$

$$\Rightarrow y=-2\{\cos^2 x-\frac{1}{2}\cos x\}$$

$$\Rightarrow y=-2\left\{\cos^2 x-\frac{1}{2}\cos x+\left(\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2\right\}$$

$$\Rightarrow y=\frac{1}{8}-2\left(\cos x-\frac{1}{2}\right)^2$$

For y_{max} put $\cos x = \frac{1}{2} \Rightarrow y_{max} = \frac{1}{8}$

Hint:

(i) Solution of D.E will be

$$y(\text{I.F}) = \int(\sin x)(\text{I.F})dx$$

(ii) I.F = $e^{\int 2\tan x\, dx}$

(iii) $\int \tan x dx=\ln(\sec x)+\text{C}$

(iv) $-1 \le \cos x \le 1$

Shortcut Method:

$\text{I. F.} = e^{2\int \tan x dx}=\sec^2 x$

$y.\sec^2 x=\int \sin x.\sec^2 x dx$

$\Rightarrow y\sec^2 x=\sec x+\text{C}$

As $y\left(\frac{\pi}{3}\right)=0\Rightarrow \text{C}=-2$

$\Rightarrow y\left(\sec^2 x\right)=\sec x-2$

$\Rightarrow y=\cos x-2\cos^2 x$

$\Rightarrow y = t-2t^2\ ;\ t=\cos x$

$\Rightarrow \frac{dy}{dt}=1-4t$, when $\frac{dy}{dt}=0\Rightarrow t=\frac{1}{4}$

$y_{max} = \frac{1}{8}$

14. Option (4) is correct.

p	q	$p\wedge q$	$p\to q$	$(p\wedge q)\to(p\to q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

$(p\wedge q)\to(p\to q)$ is tautology

> **Hint:**
> Find
> (i) $p \wedge q$
> (ii) $p \rightarrow q$
> (iii) $(p \wedge q) \rightarrow (p \rightarrow q)$

15. Option (1) is correct.

Given $A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}; i = \sqrt{-1}$ ...(i)

We have to find the solution of the system of linear equations

$$A^8 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$$ ...(ii)

Now using Equation (i)

$$A^2 = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}\begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} i^2 + i^2 & -i^2 - i^2 \\ -i^2 - i^2 & i^2 + i^2 \end{bmatrix}$$ $(\because i^2 = -1)$

$$\Rightarrow A^2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

$$\Rightarrow A^2 = 2\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow A^4 = 4\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow A^4 = 4\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$\Rightarrow A^4 = 8\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow A^8 = 64\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow A^8 = 64\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$\Rightarrow A^8 = 128\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$ ...(iii)

Now using the equation (ii) and (iii)

$$128\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$$

$$\Rightarrow 128\begin{bmatrix} x - y \\ -x + y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x - y \\ -x + y \end{bmatrix} = \frac{1}{128}\begin{bmatrix} 8 \\ 64 \end{bmatrix}$$

$$x - y = \frac{1}{16}$$ ...(iv)

$$-x + y = \frac{1}{2}$$ ...(v)

from equation (iv) and (v)

System of equation has no solution.

> **Shortcut Method:**
>
> $$A^2 = 2\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$
>
> $$A^4 = 8\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, A^8 = 128\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
>
> $$A^8\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix} \Rightarrow \begin{bmatrix} 128(x-y) \\ 128(-x+y) \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$$
>
> $\Rightarrow x - y = \frac{1}{16}$ and $-x + y = \frac{1}{2}$.
>
> Hence no solution.

16. Option (3) is correct.

Given

$$\log_{10} \sin x + \log_{10} \cos x = -1, x \in \left(0, \frac{\pi}{2}\right)$$

$$\log_{10} (\sin x \,.\, \cos x) = -1$$

$$\sin x.\, \cos x = (10)^{-1}$$

$$\sin x.\, \cos x = \frac{1}{10}$$ ...(i)

Given,

$$\log_{10} (\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1), n > 0$$

Using $\log_a a = 1$, we can write

$$2\log_{10} (\sin x + \cos x) = \frac{1}{2}(\log_{10} n - \log_{10} 10)$$

$$\log_{10} (\sin x + \cos x)^2 = \log_{10}\left(\frac{n}{10}\right)$$

$$\Rightarrow (\sin x + \cos x)^2 = \frac{n}{10}$$

$$\Rightarrow \sin^2 x + \cos^2 x + 2\sin x \cos x = \frac{n}{10}$$

$$\Rightarrow 1 + 2\sin x.\, \cos x = \frac{n}{10}$$ ...(ii)

Now using equation (i), we can write

$\Rightarrow 1 + 2\left(\frac{1}{10}\right) = \frac{n}{10}$

$\Rightarrow \quad 1+\frac{1}{5} = \frac{n}{10}$

$\Rightarrow \quad \frac{6}{5} = \frac{n}{10}$

$\Rightarrow \quad n = 12$

Hint:

$\log_a m + \log_a n = \log_a (m.n)$

$\log_a a = 1$

$\log_a b^m = m \log_a b$

$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

$\sin^2 x + \cos^2 x = 1$

Shortcut Method:

$\log_{10}(\sin x) + \log_{10}(\cos x) = -1$

$\Rightarrow \quad \sin x.\cos x = 10^{-1}$

$\log_{10}(\sin x + \cos x) = \frac{1}{2}\left(\log_{10}^{n} - 1\right)$

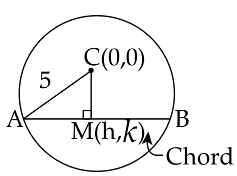
$\Rightarrow \quad 1 + 2\sin x \cos x = \frac{n}{10} \Rightarrow n = 12.$

17. Option (4) is correct.

Given equation of circle is $x^2 + y^2 = 25$.

C (0, 0) and radius $r = 5$

Let the mid pint of the chord of the circle $x^2 + y^2 = 25$ be M (h, k)



In the above figure

AM = MB and AC = 5 (radius)

Slope of MC, $m_{MC} = \frac{k-0}{h-0}$

$\Rightarrow \quad m_{MC} = \frac{k}{h}$

Let slope of AB be m_{AB}

Then, $\quad m_{AB}.\, m_{MC} = -1 \qquad \because MC \perp AB$

$m_{AB} = \frac{-h}{k}$

Equation of chord AB

$y - k = m_{AB}(x - h)$

$\Rightarrow \quad y - k = \frac{-h}{k}(x-h)$

$\Rightarrow \quad ky = -hx + h^2 + k^2$

$\Rightarrow \quad y = \left(\frac{-h}{k}\right)x + \left(\frac{h^2+k^2}{k}\right) \quad$ ...(i)

Since, the equation (i) is the tangent to the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$. ...(ii)

If $y = mx + c$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $(a < b)$ then

$c^2 = a^2 m^2 - b^2$

From equation (i) and (ii),

$\left(\frac{h^2+k^2}{k}\right)^2 = (9)\left(\frac{-h}{k}\right)^2 - (16)$

$\Rightarrow \left(\frac{h^2+k^2}{k}\right)^2 = \frac{9h^2 - 16k^2}{k^2}$

$\Rightarrow \left(h^2+k^2\right)^2 = 9h^2 - 16k^2$

Replace h and k *by* x and y

$(x^2 + y^2)^2 = 9x^2 - 16y^2$

$(x^2 + y^2)^2 = 9x^2 + 16y^2 = 0$

Hint:

(i) Let A (x_1, y_1) and B (x_2, y_2) then

Equation of line $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

and

$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$

(ii) If two lines are perpendicular to each other then product of slopes of these two lines will be equal to (–1)

(iii) If line $y = mx + c$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, (a < b)$ then

$c^2 = a^2 m^2 - b^2$

Shortcut Method:

Tangent to the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$ is

$y = mx \pm\sqrt{9m^2-16}$ ...(i)

Given that it is a chord of circle $x^2 + y^2 = 25$ with midpoint (h, k)

$T = S_1$

$hx + ky = h^2 + k^2$

$\Rightarrow \quad y = \frac{-hx}{k}+\left(\frac{h^2+k^2}{k}\right)$ ...(ii)

Using equation (i) and (ii)

$m = \frac{-h}{k}$ and $\sqrt{9m^2-16} = \frac{h^2+k^2}{k}$

$\Rightarrow 9\frac{h^2}{k^2}-16=\frac{\left(h^2+k^2\right)^2}{k^2}$

$\Rightarrow 9x^2-16y^2=(x^2+y^2)^2$

18. Option (1) is correct.

Given that

$$(1 - x + x^3)^n = \sum_{j=0}^{3n} a_j x^j$$

$\Rightarrow \quad (1 - x + x^3)^n = a_0 + a_1 x + a_2x^2 + a_3 x^3 + \ldots\ldots + a_{3n} x^{3n}$ ...(i)

We have to find the value of

$$\sum_{j=0}^{\left[\frac{3n}{2}\right]} a_{2j} + 4\sum_{j=0}^{\left[\frac{3n-1}{2}\right]} a_{2j+1}$$

Here, $\sum_{j=0}^{\left[\frac{3n}{2}\right]} a_{2j} = a_0 + a_2 + a_4 + \ldots$..and

$$\sum_{j=0}^{\left[\frac{3n-1}{2}\right]} a_{2j+1} = a_1 + a_3 + a_5 + \ldots..$$

Put $x = 1$ in equation (i)

$1 = a_0 + a_1 + a_2 + a + \ldots + a_{3n}$ ...(ii)

Put $x = -1$ in equation (i)

$1 = a_0 - a_1 + a_2 - a_3 + \ldots\ldots(-1)^{3n} a^{3n}$ ...(iii)

After adding (ii) and (iii) we get

$2 = 2(a_0 + a_2 + a_4 + \ldots\ldots)$

$\Rightarrow a_0 + a_2 + a_4 + \ldots.. = 1$

i.e. $\sum_{j=0}^{\left[\frac{3n}{2}\right]} a_{2j} = 1$ ...(iv)

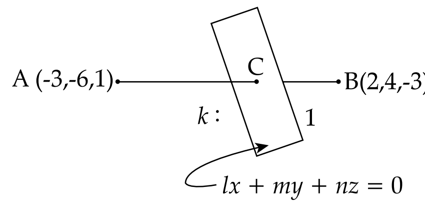
and $a_1 + a_3 + a_5 + \ldots. = 0$

i.e. $4\sum_{j=0}^{\left[\frac{3n-1}{2}\right]} a_{2j+1} = 0$ ...(v)

Add equation (iv) and (v)

$$\Rightarrow \sum_{j=0}^{\left[\frac{3n}{2}\right]} a_{2j+1} + 4\sum_{j=0}^{\left[\frac{3n-1}{2}\right]} a_{2j+1} = 1$$

19. Option (2) is correct.



Given; $A(-3, -6, 1)$, $B(2, 4, -3)$

Plane P divide the line segment AB in the ratio k: 1

$C\left(\frac{2k-3}{k+1},\frac{4k-6}{k+1},\frac{-3k+1}{k+1}\right)$

Equation of line $\frac{1-x}{1}=\frac{y+4}{2}=\frac{z+2}{3}$ satisfied to the plane then, $lx + my + nz = 0$

$\Rightarrow l(-1) + m(2) + n(3) = 0$

$\Rightarrow -l + 2m + 3n = 0$ ...(ii)

Since, $lx + my + nz = 0$ also satisfy point $(1, -4, -2)$

Then, $l - 4m - 2n = 0$ ...(iii)

Now using (ii) and (iii)

$n = 2m$

$l = 8m$

$\frac{l}{8}=\frac{m}{1}=\frac{n}{2}$

$l : m : n = 8 : 1 : 2$

Equation of plane will be $8x + y + 2z = 0$

Point C will satisfy $8x + y + 2z = 0$, then

$$8\left(\frac{2k-3}{k+1}\right)+\left(\frac{4k-6}{k+1}\right)+2\left(\frac{-3k+1}{k+1}\right)=0$$

$\Rightarrow 16k - 24 + 4k - 6 - 6k + 2 = 0$

$\Rightarrow k = 2$

Hint:

(i) let plane cuts segment AB at C then

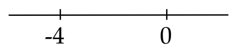
$$C\left(\frac{2k-3}{k+1}\right)+\left(\frac{4k-6}{k+1}\right)+2\left(\frac{-3k+1}{k+1}\right)=0$$

(ii) Point C will satisfy the plane .

20. Option (1) is correct.

Given,

$$(|x|-3)(|x+4|) = 6$$



Case I: when $x < -4$

Since, $|x| = \begin{cases} x; x > 0 \\ -x; x < 0 \end{cases}$

$\Rightarrow \quad (-x-3)(-x-4) = 6$

$\Rightarrow \quad (x+3)(x+4) = 6$

$\Rightarrow \quad x^2 + 7x + 6 = 0$

$\Rightarrow \quad (x+6)(x+1) = 0$

$\Rightarrow \quad x = -1$ and -6

As $x < -4$ so for this case $x = \{-6\}$

Case II: when $x \in [-4, 0)$

Using $|x| = \begin{cases} x;\ x \geq 0 \\ -x;\ x < 0 \end{cases}$

$(-x-3)(x+4) = 6$

$\Rightarrow \quad x^2 + 7x + 18 = 0$

$\Rightarrow$ No solution ($D < 0$)

So, for this case $x \in \phi$

Case III:

When $x \geq 0$

$(x-3)(x+4) = 6$

$\Rightarrow \quad x^2 + x - 18 = 0$

$\Rightarrow \quad x = \dfrac{-1 \pm \sqrt{1+72}}{2}$

As $x \geq 0$ so for this case $x = \left\{\dfrac{\sqrt{73}-1}{2}\right\}$

So, final solution of the given equation are

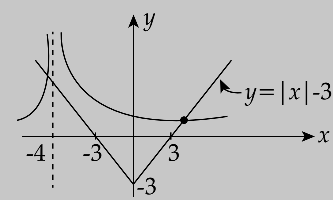
$x \in \left\{-6, \dfrac{\sqrt{73}-1}{2}\right\}$

Hence number of solution will be 2.

Shortcut Method:

$(|x|-3)(|x+4|) = 6$

$\Rightarrow \quad |x| - 3 = \dfrac{6}{|x+4|}$



From the above figure, we see that here 2 intersection are present so number of solution will be two.

Section B

21. Correct answer is [1].

Given that,

$f: (0, 2) \rightarrow R$

$f(x) = \log_2\left(1+\tan\left(\dfrac{\pi x}{4}\right)\right)$ ...(i)

We have to find the value of

$$\lim_{n\to\infty}\frac{2}{n}\left(f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+.....+f(1)\right)$$

Let,

$$L = \lim_{n\to\infty}\left(2\left(\sum_{r=1}^{n}\left(\frac{1}{n}\right)f\left(\frac{r}{n}\right)\right)\right)$$

By replacing, $\dfrac{1}{n} \to dx$, $\dfrac{r}{n} \to x$, $\lim\limits_{n\to\infty}\sum \to \int$

Lower limit $= \lim\limits_{n\to\infty}\left(\dfrac{1}{n}\right) = 0$

Upper limit $= \lim\limits_{n\to\infty}\left(\dfrac{n}{n}\right) = 1$

We get,

$L = 2\int_0^1 f(x)dx$ ...(ii)

Now using equation (i) and (ii)

$$L = 2\int_0^1 \log_2\left(1+\tan\left(\frac{\pi x}{4}\right)\right)dx$$

Using, $\log_a x = \dfrac{\ln x}{\ln a}$ we can write

$$L = 2\int_0^1 \left(\frac{\ln\left(1+\tan\left(\frac{\pi x}{4}\right)\right)}{\ln 2}\right)dx$$

$$\Rightarrow L = \frac{2}{\ln 2}\int_0^1 \ln\left(1+\tan\left(\frac{\pi x}{4}\right)\right)dx \quad \text{...(iii)}$$

Now using the property

$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx,$

We can write

$$\Rightarrow L = \frac{2}{\ln 2}\int_0^1 \ln\left(1+\tan\left(\frac{\pi(1-x)}{4}\right)\right)dx$$

$$\Rightarrow L = \frac{2}{\ln 2}\int_0^1 \ln\left(1+\tan\left(\frac{\pi}{4}-\frac{\pi x}{4}\right)\right)dx$$

$$\Rightarrow L = \frac{2}{\ln 2}\int_0^1 \ln\left(1+\frac{\tan\frac{\pi}{4}-\tan\frac{\pi x}{4}}{1+\tan\frac{\pi}{4}.\tan\frac{\pi x}{4}}\right)dx$$

$$\Rightarrow L = \frac{2}{\ln 2}\int_0^1 \ln\left(1+\frac{1-\tan\frac{\pi x}{4}}{1+\tan\frac{\pi x}{4}}\right)dx$$

$$\Rightarrow L = \frac{2}{\ln 2}\int_0^1 \ln\left(\frac{2}{1+\tan\frac{\pi x}{4}}\right)dx \quad \text{...(iv)}$$

After adding equation (iii) and (iv), we get

$$\Rightarrow 2L = \frac{2}{\ln 2}\int_0^1 \left\{\ln\left(1+\tan\frac{\pi x}{4}\right)+\ln\left(\frac{2}{1+\tan\frac{\pi x}{4}}\right)\right\}dx$$

$$\Rightarrow L = \frac{1}{\ln 2}\int_0^1 \ln 2dx$$

$$\Rightarrow L = \int_0^1 dx$$

$$\Rightarrow L = (x)_0^1$$

$\Rightarrow L = 1 - 0$

$\Rightarrow L = 1$

Hint:

(i) Use changing limit as a sum into definite integration for this replace

$\frac{1}{n}$ by dx

$\frac{r}{n}$ by x

$\lim_{n\to\infty} \sum$ by $\int$

Lower limit $= \lim_{n\to\infty} \frac{r_{min}}{n}$

Upper limit $= \lim_{n\to\infty} \frac{r_{max}}{n}$

(ii) Use logarithmic properties like $\log_a m + \log_a n = \log_a (mn)$

$$\log_a b = \frac{\log_e b}{\log_e a} = \frac{\ln b}{\ln a}$$

(iii) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

(iv) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Shortcut Method:

$$L = \lim_{n\to\infty} \frac{2}{n}\left(f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+.....+f(1)\right)$$

$$\Rightarrow L = \frac{2}{\ln 2}\int_0^1 \ln(1+\tan\frac{\pi x}{4})dx$$

$$\Rightarrow 2L = \frac{2}{\ln 2}\int_0^1 \ln(2)dx$$

$$\Rightarrow L = \int_0^1 dx \Rightarrow L = 1.$$

22. Correct answer is [766].

Given that order of matrix A is 3×3 .

$$\text{Let } A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3\times 3}$$

$$\text{Then } A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}_{3\times 3}$$

$$AA^T = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$\Rightarrow AA^T = \begin{bmatrix} a^2+b^2+c^2 & ad++be+cf & ag+bh+ci \\ ad+be+fc & d^2+e^2+f^2 & dg+eh+fi \\ ga+hb+ci & gd+he+fi & g^2+h^2+i^2 \end{bmatrix}$$

Now, sum of all the diagonal entries of $AA^T = 9$

$\Rightarrow T_r(AA^T) = 9$

$\Rightarrow a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2 + i^2 = 9$ ...(i)

Given that matrix A having entries from the set {0, 1, 2, 3}

Hence $a, b, c, d, e, f, g, h, i, \in \{0, 1, 2, 3\}$

S.N.	Case	Number of matrices
1	All $\to$ 1s	$\frac{9!}{9!} = 1$
2	One $\to$ 3, Remaining $\to$ 0s	$\frac{9!}{1!8!} = 9$
3	One $\to$ 2 Five $\to$ 1s Three $\to$ 0s	$\frac{9!}{1!5!3!} = 504$
4	Two $\to$ 2s One $\to$ 1 Six $\to$ 0s	$\frac{9!}{2!6!} = 252$

So total number of ways

$$= \frac{9!}{9!} + \frac{9!}{1!8!} + \frac{9!}{1!5!3!} + \frac{9!}{2!6!}$$

= 1 + 9 + 504 + 252 = 766.

23. Correct answer is [16].

Given that

$f: R \to R$ is continuous if

$f(x) + f(x+1) = 2$ ...(i)

Replace x by $x + 1$

$f(x+1) + f(x+2) = 2$ ...(ii)

Now, from equation (ii) – (i), we have

$f(x+1) + f(x+2) - f(x) - f(x+1) = 2 - 2$

$\Rightarrow f(x+2) - f(x) = 0$

$\Rightarrow f(x) = f(x+2)$ ...(iii)

If $f(x) = f(x + T)$ then $f(x)$ will be a periodic function with period T.

Using the concept we can say that given function is periodic with period 2.

Now,

$$I_1 = \int_0^8 f(x)dx$$

$$\Rightarrow I_1 = \int_0^{2\times4} f(x)dx$$

$$\Rightarrow I_1 = 4\int_0^2 f(x)dx \quad \text{...(iv)}$$

Also

$$I_2 = \int_{-1}^3 f(x)dx$$

$$\Rightarrow I_2 = \int_0^4 f(x+1)dx$$

Using Equation (i)

$$I_2 = \int_0^4 (2 - f(x))dx$$

$$\Rightarrow I_2 = \int_0^4 2dx - \int_0^4 f(x)dx$$

$$\Rightarrow I_2 = 2\int_0^4 dx - \int_0^4 f(x)dx$$

$$\Rightarrow I_2 = 2(x)_0^4 - \int_0^{2\times2} f(x)dx$$

$$\Rightarrow I_2 = 2(4-0) - 2\int_0^2 f(x)dx$$

Now Using Equation (iv)

$$\Rightarrow I_2 = 8 - 2\left(\frac{I_1}{4}\right)$$

$$\Rightarrow I_2 = 8 - \frac{I_1}{2}$$

$\Rightarrow 2I_2 = 16 - I_1$

$\Rightarrow I_1 + 2I_2 = 16$

Hint:

If $f(x+T) = f(x)$, then $f(x)$ will be a periodic function with period T(T > 0).

$$\int_0^{nT} f(x)dx = n\int_0^T f(x)dx,$$

where T is period of $f(x)$.

Shortcut Method:

$f(x) = f(x + 1) = 2$

$\Rightarrow f(x)$ is periodic function with period 2.

$$I_1 = \int_0^8 f(x)dx = 4\int_0^2 f(x)dx$$

$$\Rightarrow \quad I_1 = 4\int_0^1 (f(x) + f(1+x))dx$$

$\Rightarrow \quad I_1 = 8$

Similarly

$I_2 = 4$

$I_1 + 2I_2 = 16.$

24. Correct answer is [3].

Given set = {11, 8, 21, 16, 26, 32, 4}

According to the problem,

G. P.: 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192

Last term of G.P. will be 8192, because given that the last term of this series is the maximum four digit number.

Now

A. P.: 11, 16, 21, 26, 31, 36, …, 251, 256, 261, …. 4091, 4096….

From the above A.P. and G.P., Common terms will be 16, 256, 4096 only.

Shortcut Method:

A.P. : 11, 16, 21, 26, … 4096

G.P. : 4, 8, 16, 32, …, 4096, 8192

So, common terms will be 16, 256, 4096 only.

25. Correct answer is [406].

Given,

$$y(x)=\int_0^x \left(2t^2-15t+10\right)dt$$

Apply Leibnitz theorem

$$H(x)=\int_{f_2(x)}^{f_1(x)} g(t)dt$$

$$H'(x)=g\left(f_1(x)\right)f_1'(x)-g\left(f_2(x)\right)f_2'(x)$$

Using the Leibnitz theorem, we can write

$$y'(x)=\left(2x^2-15x+10\right)\frac{d(x)}{dx}-(10)\frac{d(0)}{dx}$$

$$\Rightarrow y'(x)=2x^2-15x+10$$

$y'(x)$ at point P (a, b).

$$y'(a)=2a^2-15a+10 \quad \text{...(i)}$$

Normal is parallel to the line $x + 3y = -5$,

So slope of normal will be equal to $-\frac{1}{3}$.

Let $m_N=-\frac{1}{3}$

$m_T = 3$; $(m_N \cdot m_T = -1)$ ...(ii)

$y'(a)=m_T$

Now using equation (i) and (ii)

$$2a^2 - 15a + 10 = 3$$

$$\Rightarrow (a-7)(2a-1)=0$$

$$\Rightarrow a=7,\ \frac{1}{2}$$

So, $a = 7$, $a=\frac{1}{2}$ rejected because $a > 1$ ...(iii)

For value of b:

$b = y(a)$

$\Rightarrow b = y(7)$

$$\Rightarrow b=\int_0^7\left(2t^2-15t+10\right)dt$$

$$\Rightarrow b=\left(\frac{2t^3}{3}-\frac{15t^2}{2}+10t\right)_0^7$$

$$\Rightarrow b=\left(\frac{2}{3}\right)(7)^3-\frac{15}{2}(7)^2+10(7)-0$$

$$\Rightarrow b=\frac{(2)^2(7)^3-(15)(7)^2(3)+(10)(7)(6)}{6}$$

$\Rightarrow 6b = (4 \times 343) - (15)(49)(3) + 420$

$\Rightarrow 6b = 1372 - 2205 + 420$

$\Rightarrow 6b = 1792 - 2205$

$\Rightarrow 6b = -413$ ...(iv)

Now using equation (iii) and (iv)

$|a+6b| = |7-413|$

$\Rightarrow |a+6b| = |-406| \quad \because |x| = \begin{cases} x; & x \ge 0 \\ -x; & x < 0 \end{cases}$

$\Rightarrow |a+6b| = 406$

Hint:

(i) If slope of line L_1 is m_1 and slope of line L_2 is m_2. Now if L_1 and L_2 are parallel then $m_1 = m_2$ and if L_1 is perpendicular to L_2 then $m_1 \cdot m_2 = -1$.

(ii) Leibnitz Rule:

$$\frac{d}{dx}\left(\int_{f_2(x)}^{f_1(x)} g(t)dt\right)=g\left(f_1(x)\right)\frac{d}{dx}\left(f_1(x)\right)-g\left(f_2(x)\right)\frac{d}{dx}\left(f_2(x)\right)$$

(iii) $|x| = \begin{cases} x; & x \ge 0 \\ -x; & x < 0 \end{cases}$

26. Correct answer is [4].

Given that

$$\lim_{x\to 0}\frac{ae^x-b\cos x+ce^{-x}}{x\sin x}=2 \quad \text{...(i)}$$

Since, limit exists, therefore one of the indeterminant form will be present.

Now for indeterminant $\left(\frac{0}{0}\right)$ form

Since, $e^x = 1 + x + \frac{x^2}{2!} + \ldots$, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots$,

$$e^{-x} = 1 - x + \frac{x^2}{2!} \ldots$$

Using equation (i)

$$\lim_{x\to 0} \frac{a\left(1 + x + \frac{x^2}{2!} + \ldots\right) - b\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + c\left(1 - x + \frac{x^2}{2!} - \ldots\right)}{x\left(\frac{\sin x}{x}\right)(x)} = 2$$

$$\lim_{x\to 0} \frac{(a-b+c) + x(a-c) + x^2\left(\frac{a}{2} + \frac{b}{2} + \frac{c}{2}\right)}{x^2\left(\frac{\sin x}{x}\right)} = \mathbf{2}$$

Comparing LHS and RHS

$(a - b + c) = 0$, $(a - c) = 0$, $\frac{a+b+c}{2} = 2$

$\Rightarrow a + b + c = 4$

Shortcut Method:

$$\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2$$

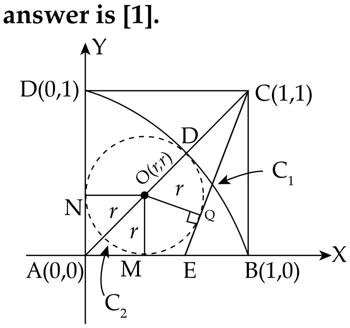
$$\Rightarrow \lim_{x\to 0} \frac{(a-b+c) + (a-c)x + \left(\frac{a+b+c}{2}\right)x^2 + \ldots}{x^2\left(\frac{\sin x}{x}\right)} = 2$$

$\Rightarrow a - b + c = 0$

$a - c = 0$

$\frac{a+b+c}{2} = 2 \Rightarrow a + b + c = 4.$

27. Correct answer is [1].



C_1: center = A(0, 0)

Radius = 1 unit = AD

OD = AM = MO = $r \Rightarrow$ AO = $\sqrt{2}\,r$

$\because$ AD = 1

AO + OD = 1

$\Rightarrow \sqrt{2}r + r = 1$

$\Rightarrow r = \frac{1}{\sqrt{2}+1}$

$\Rightarrow r = \sqrt{2} - 1$

So for circle C_2:

Center is (r, r) and

Radius is $\sqrt{2} - 1$.

So, equation circle will be

$$(x-r)^2 + (y-r)^2 = \left(\sqrt{2}-1\right)^2$$

Let slope of line, which passes through point C(1, 1) and E is m. Then equation of line will be

$y - 1 = m(x - 1)$;

$m > 0$ (From figure)

$mx - y + 1 - m = 0$ ...(i)

Line CE will be tangent to circle C_2 if

OQ = r; where OQ is perpendicular distance of point O(r, r) from the line CE.

Hence,

$$\left|\frac{mr - r + 1 - m}{\sqrt{m^2+1}}\right| = \sqrt{2} - 1;\ r = \sqrt{2} - 1$$

$$\Rightarrow \left|\frac{(r-1)(m-1)}{\sqrt{m^2+1}}\right| = \sqrt{2} - 1$$

As $r = \sqrt{2} - 1$

$$\Rightarrow \left|\frac{\left(\sqrt{2}-2\right)(m-1)}{\sqrt{m^2+1}}\right| = \sqrt{2} - 1$$

$$\Rightarrow \left|\frac{\sqrt{2}(m-1)}{\sqrt{m^2+1}}\right| = 1$$

Squaring both the sides

$$\Rightarrow \frac{2(m-1)^2}{m^2+1} = 1$$

$\Rightarrow m^2 - 4m + 1 = 0$

$\Rightarrow m = 2 \pm \sqrt{3}$

As $m > 0$, $2 - \sqrt{3}$ rejected

$m = 2 + \sqrt{3}$

Now using equation (i)

$$\left(2+\sqrt{3}\right)x - y + 1 - \left(2+\sqrt{3}\right) = 0$$

For E, $y = 0$

$$x = \frac{\sqrt{3}+1}{2+\sqrt{3}}$$

$\Rightarrow x = \sqrt{3} - 1$

E ($\sqrt{3}$ – 1, 0) and B(1, 0)

EB = OB – OE

EB = 1– ($\sqrt{3}$ – 1)

EB = 2 – $\sqrt{3}$

$\alpha + \sqrt{3}\beta = 2 + \sqrt{3}(-1)$; EB = $\alpha + \sqrt{3}\beta$

After comparing we get

$\alpha = 2, \beta = -1$

$\alpha + \beta = 1.$

Hint:

(i) Assume radius of circle C_2 as r. Now with the help of radius of circle C_1 Find value of r.

(ii) A line $y = mx + c$ will be tangent to a circle $(x - x_1)^2 + (y - y_1)^2 = r^2$

if $\left|\frac{mx_1 - y_1 + c}{\sqrt{1+m^2}}\right| = r$

(iii) Find the equation of line CE.

(iv) EB = OB – OE.

AP = OP = $r \Rightarrow$ AO = $r\sqrt{2}$

AS = 1 $\Rightarrow$ AO + OS = 1 $\Rightarrow r\sqrt{2} + r = 1$

$\Rightarrow r = \sqrt{2} - 1$

Now OC = $2\sqrt{2} - 2 \Rightarrow$ OC = $2(\sqrt{2} - 1)$

In Δ COQ:

$$\sin\theta = \frac{\sqrt{2}-1}{2\left(\sqrt{2}-1\right)} \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

In Δ ACE

$$\Rightarrow \quad \frac{AE}{\sin\theta} = \frac{AC}{\sin 105^\circ};\ \theta = \frac{\pi}{6}$$

$$\Rightarrow AE = \frac{\sqrt{2}}{2\times\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}$$

$\Rightarrow$ AE = $\sqrt{3} - 1$

EB = AB – AE

$\Rightarrow$ EB = 1 – $\sqrt{3}$ + 1 = 2 – $\sqrt{3}$

$\Rightarrow \alpha + \sqrt{3}\beta = 2 - \sqrt{3} \Rightarrow \alpha + \beta = 1.$

28. Correct answer is [4].

Given that

$$\omega = z\bar{z} - 2z + 2 \text{ and } \left|\frac{z+i}{z-3i}\right| = 1 \qquad \text{...(i)}$$

Using $\left|\frac{z_1}{z_2}\right| = 1 \Rightarrow |z_1| = |z_2|$,

We can write $\left|\frac{z+i}{z-3i}\right| = 1$ as,

$$|z+i| = |z-3i| \qquad \text{...(ii)}$$

Put $z = x + iy$ in equation (ii)

$$|x+iy+i| = |x+iy-3i|$$

$$\Rightarrow |x+(y+1)i| = |x+(y-3)i| \qquad \text{...(iii)}$$

Since, $|z| = \sqrt{x^2+y^2}$,

So from equation (iii)

$$\sqrt{x^2+(y+1)^2} = \sqrt{x^2+(y-3)^2}$$

$\Rightarrow x^2 + y^2 + 2y + 1 = x^2 + y^2 - 6y + 9$

$\Rightarrow 8y = 8 \quad \Rightarrow y = 1$ ...(iv)

Now using equation (i)

$$\omega = (x+iy)\left(\overline{x+iy}\right) - 2(x+iy) + 2$$

using $z.\bar{z} = |z|^2$, we can write

$$\omega = x^2 + y^2 - 2(x+iy) + 2$$

$$\Rightarrow \omega = \left(x^2+y^2-2x+2\right) + 2yi$$

$$\omega = \left(x^2-2x+3\right) - 2i;\ y = 1 \qquad \text{...(v)}$$

Now,

$$\text{Re}(\omega) = x^2 - 2x + 3$$

$$\Rightarrow \text{Re}(\omega) = 2 + (x-1)^2$$

Re(ω) will be minimum when $x = 1$.

Hence $z = 1 + i$; $x = y = 1$

Now using equation (v)

$$\omega = 1 - 2 + 3 - 2i \Rightarrow \omega = 2 - 2i$$

$$\Rightarrow \omega = 2(1-i) = \omega \ \Rightarrow\ 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$$

$$\omega = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)i\right)$$

$$\Rightarrow \omega = 2\sqrt{2}\left(\cos\frac{\pi}{4} + \sin\left(-\frac{\pi}{4}\right)i\right)$$

$$\omega = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{4}\right)i\right)$$

$$\Rightarrow \omega = 2\sqrt{2}e^{i\left(-\frac{\pi}{4}\right)} \Rightarrow \omega^n = \left(2\sqrt{2}\right)^n e^{i\left(-\frac{n\pi}{4}\right)}$$

ω^n is real and minimum when $n = 4$.

Shortcut Method:

Let $z = x+iy;\ \bar{z} = x-iy;$

$$z.\bar{z} = x^2+y^2; |z| = \sqrt{x^2+y^2}$$

$$\omega = z.\bar{z}-2z+2$$

$$\Rightarrow \omega = x^2+y^2+2-2x-2yi \quad \text{...(i)}$$

$$\left|\frac{z+i}{z-3i}\right| = 1 \Rightarrow |z+i| = |z-3i| \Rightarrow y = 1 \quad \text{...(ii)}$$

So, $\omega = x^2-2x+3-2i$

$$\Rightarrow \text{Re}(\omega) = x^2-2x+3 = 2+(x-1)^2$$

Re(ω) min at $x = 1$.

$z = 1 + i$

$$\omega = 2\sqrt{2}e^{i\left(-\frac{\pi}{4}\right)} \Rightarrow \omega^n = \left(2\sqrt{2}\right)^n e^{\left(\frac{-n\pi}{4}\right)i}$$

ω^n is real and minimum when $n = 4$.

29. Correct answer is [36].

Given that

$$P = \begin{bmatrix} -30 & 20 & 56 \\ 90 & 140 & 112 \\ 120 & 60 & 14 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 7 & \omega^2 \\ -1 & -\omega & 1 \\ 0 & -\omega & -\omega+1 \end{bmatrix}; \omega = \frac{-1+i\sqrt{3}}{2}$$

Let $M = \left(P^{-1}AP - I_3\right)^2$

$$\Rightarrow M = \left(P^{-1}AP\right)^2 + (I_3)^2 - 2\left(P^{-1}AP\right)(I_3)$$

$$M = \left(P^{-1}AP\right)^2 + (I_3)^2 - 2\left(P^{-1}AP\right)$$

$$M = P^{-1}A^2P + I_3 - 2P^{-1}AP$$

$$PM = A^2P + PI_3 - 2AP$$

$$\Rightarrow PM = (A^2 + I_3 - 2A)P$$

$$PM = (A^2 + (I_3)^2 - 2A\,I_3)P$$

$$\Rightarrow PM = (A - I_3)^2 P$$

$$\text{Det(PM)} = \text{Det}\left((A-I_3)^2 P\right)$$

$$(\text{Det } P)(\text{Det } M) = \left(\text{Det}(A-I_3)^2\right)(\text{Det } P)$$

$$\text{Det } M = \text{Det}(A-I_3)^2$$

$$\text{Now, } A - I_3 = \begin{bmatrix} 2 & 7 & \omega^2 \\ -1 & -\omega & 1 \\ 0 & -\omega & -\omega+1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - I_3 = \begin{bmatrix} 1 & 7 & \omega^2 \\ -1 & -\omega-1 & 1 \\ 0 & -\omega & -\omega \end{bmatrix}$$

$$\text{Det}(A-I_3) = 1\left(\omega^2+\omega+\omega\right) - 7(\omega-0) + \omega^2(\omega-0)$$

$$\Rightarrow \text{Det}(A-I_3) = \omega^2 + 2\omega - 7\omega + \omega^3$$

$$\Rightarrow \text{Det}(A-I_3) = \omega^3 + \omega^2 - 5\omega$$

$$\Rightarrow \text{Det}(A-I_3) = -6\omega$$

$$\Rightarrow \text{Det}(A-I_3)^2 = 36\omega^2 \Rightarrow \alpha\omega^2 = 36\omega^2$$

$$\Rightarrow \alpha = 36.$$

Hint:

(i) $\left(P^{-1}AP - I_3\right)^2 = P^{-1}A^2P + I_3 - 2P^{-1}AP$

(ii) $P\left(P^{-1}AP - I_3\right)^2 = (A - I_3)^2 P$

(iii) $\omega^3 + \omega^2 - 5\omega = -6\omega$.

30. Correct answer is [2].

Given

$$\frac{dy}{dx} = 2(x+1) \quad \Rightarrow dy = (2x+2)dx$$

Now, Integrating above,

$$\int dy = \int (2x+2)dx \Rightarrow y = \left(\frac{x^2}{2}\right)2 + 2x + c$$

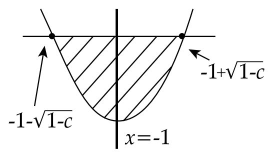
$$\Rightarrow y = x^2 + 2x + c \quad \text{...(i)}$$

Area bounded by $y = y(x)$ and x-axis is $\frac{4\sqrt{8}}{3}$.

Using equation (i), at x-axis ($y = 0$).

$$x^2 + 2x + c = 0$$

$$x = \frac{-2 \pm \sqrt{4-4c}}{2} \Rightarrow x = -1 \pm \sqrt{1-c}$$



$$A = 2\int_{-1}^{-1+\sqrt{1-c}} \left\{-(x+1)^2 - c + 1\right\} dx = \frac{4\sqrt{8}}{3}$$

$$\Rightarrow \left[\frac{-(x+1)^3}{3} - cx + x\right]_{-1}^{-1+\sqrt{1-c}} = \frac{2\sqrt{8}}{3}$$

$$\Rightarrow \frac{-\left(\sqrt{1-c}\right)^3}{3} - c\left(-1+\sqrt{1-c}\right) + \left(-1+\sqrt{1-c}\right) - c + 1 = \frac{2\sqrt{8}}{3}$$

$$\Rightarrow -\left(\sqrt{1-c}\right)^3 + 3c - 3c\sqrt{1-c} - 3 + 3\sqrt{1-c} - 3c + 3 = 2\sqrt{8}$$

$$\Rightarrow -\left(\sqrt{1-c}\right)^3 - 3c\sqrt{1-c} + 3\sqrt{1-c} = 2\sqrt{8}$$

$$\Rightarrow c = -1$$

Now using Equation (i)

$$y = y(x) = x^2 + 2x - 1$$

$$\Rightarrow y(1) = 2$$

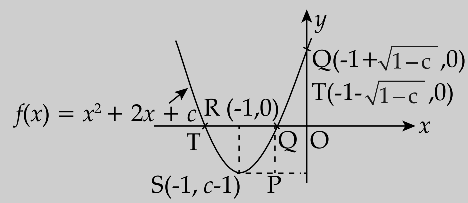
Hint:

(i) First simplify given differential equation to get y.

(ii) Find value of unknown constant (c) with the help of given area.

(iii) Replace value of c in y and find $y(1)$.

Shortcut Method:



Area of rectangle PQRS $= \left|(c-1)\left(\sqrt{1-c}\right)\right|$

Now, $\frac{4\sqrt{8}}{3} = 2\left(\frac{2}{3}(1-c)^{\frac{3}{2}}\right) \Rightarrow c = -1$

$f(x) = x^2 + 2x - 1$

$f(1) = 2.$

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