JEE (Main) MATHEMATICS SOLVED PAPER

2023

08^{th} April Shift 1

Section A

Q. 1. The area of the region $\{(x, y): x^2 \le y \le 8 - x^2, y \le 7\}$ is

(1) 24 **(2)** 21 **(3)** 20 **(4)** 18

Q. 2. Let $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$.. If P^T

$Q^{2007} P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $2a + b - 3c - 4d$ equal to

(1) 2004 **(2)** 2007 **(3)** 2005 **(4)** 2006

Q. 3. Negation of $(p \to q) \to (q \to p)$ is

(1) $(\neg q) \wedge p$ **(2)** $p \vee (\sim q)$
(3) $(\sim p) \vee q$ **(4)** $q \wedge (\sim p)$

Q. 4. Let C(α, β) be the circumcenter of the triangle formed by the lines
$4x + 3y = 69$, $4y - 3x = 17$ and $x + 7y = 61$.
Then $(\alpha - \beta)^2 + \alpha + \beta$ is equal to

(1) 18 **(2)** 15 **(3)** 16 **(4)** 17

Q. 5. Let α, β, γ, be the three roots of the equation $x^3 + bx + c = 0$. If $\beta\gamma = 1 = -\alpha$, then $b^3 + 2c^3 - 3\alpha^3 - 6\beta^3 - 8\gamma^3$ is equal to

(1) $\frac{155}{8}$ **(2)** 21 **(3)** 19 **(4)** $\frac{169}{8}$

Q. 6. Let the number of elements in sets A and B be five and two respectively. Then the number of subsets of A × B each having at least 3 and at most 6 elements is:

(1) 752 **(2)** 772 **(3)** 782 **(4)** 792

Q. 7. If the coefficients of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio 1 : 5 : 20, then the coefficient of the fourth term is

(1) 5481 **(2)** 3654 **(3)** 2436 **(4)** 1817

Q. 8. Let R be the focus of the parabola $y^2 = 20x$ and the line $y = mx + c$ intersect the parabola at two points P and Q.
Let the point G(10, 10) be the centroid of the triangle PQR. If $c - m = 6$, then $(PQ)^2$ is

(1) 325 **(2)** 346 **(3)** 296 **(4)** 317

Q. 9. Let $S_K = \frac{1+2+\ldots+K}{K}$ and $\sum_{j=1}^{n} S_j^2 = \frac{n}{A}$ $(Bn^2 + Cn + D)$, where A, B, C, D ∈ N and A has least value. Then

(1) A + B is divisible by D
(2) A + B = 5 (D –C)
(3) A + C + D is not divisible by B
(4) A + B + D is divisible by 5

Q. 10. The shortest distance between the lines $\frac{x-4}{4} = \frac{y+2}{5} = \frac{z+3}{3}$ and $\frac{x-1}{3} = \frac{y-3}{4} = \frac{z-4}{2}$ is

(1) $2\sqrt{6}$ **(2)** $3\sqrt{6}$ **(3)** $6\sqrt{3}$ **(4)** $6\sqrt{2}$

Q. 11. The number of arrangements of the letters of the word "INDEPENDENCE" in which all the vowels always occur together is.

(1) 16800 **(2)** 14800 **(3)** 18000 **(4)** 33600

Q. 12. If the points with position vectors $\alpha\hat{i} + 10\hat{j} + 13\hat{k}$, $6\hat{i} + 11\hat{j} + 11\hat{k}, \frac{9}{2}\hat{i} + \beta\hat{j} - 8\hat{k}$ are collinear, then $(19\alpha - 6\beta)^2$ is equal to

(1) 49 **(2)** 36 **(3)** 25 **(4)** 16

Q. 13. In a bolt factory, machines A, B and C manufacture respectively 20%, 30% and 50% of the total bolts. Of their output 3, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random form the product. If the bolt drawn is found the defective, then the probability that it is manufactured by the machine C is.

(1) $\frac{5}{14}$ **(2)** $\frac{3}{7}$ **(3)** $\frac{9}{28}$ **(4)** $\frac{2}{7}$

Q. 14. If for $z = \alpha + i\beta$, $|z + 2| = z + 4(1 + i)$, then $\alpha + \beta$ and $\alpha\beta$ are the roots of the equation

(1) $x^2 + 3x - 4 = 0$ **(2)** $x^2 + 7x + 12 = 0$
(3) $x^2 + x - 12 = 0$ **(4)** $x^2 + 2x - 3 = 0$

Q. 15. $\lim_{x\to 0}\left(\left(\frac{(1-\cos^2(3x))}{\cos^3(4x)}\right)\left(\frac{\sin^3(4x)}{(\log_e(2x+1))^5}\right)\right)$ is equal to ________.

(1) 24 **(2)** 9 **(3)** 18 **(4)** 15

Q. 16. The number of ways, in which 5 girls and 7 boys can be seated at a round table so that no two girls sit together, is

(1) $7\,(720)^2$ **(2)** 720
(3) $7\,(360)^2$ **(4)** $126\,(5!)^2$

Q. 17. Let $f(x) = \frac{\sin x + \cos x - \sqrt{2}}{\sin x - \cos x}, x \in [0, \pi] - \left\{\frac{\pi}{4}\right\}$. Then $f\left(\frac{7\pi}{12}\right)f''\left(\frac{7\pi}{12}\right)$ is equal to

(1) $\frac{-2}{3}$ **(2)** $\frac{2}{9}$ **(3)** $\frac{-1}{3\sqrt{3}}$ **(4)** $\frac{2}{3\sqrt{3}}$

Q. 18. If the eqation of the plane containing the line $x + 2y + 3z - 4 = 0 = 2x + y - z + 5$ and perpendicular to the plane $\vec{r} = (\hat{i} - \hat{j}) + \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k})$ is $ax + by + cz = 4$, then $(a - b + c)$ is equal to

(1) 22 **(2)** 24 **(3)** 20 **(4)** 18

Q. 19. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. If $|\text{adj (adj (adj } 2A))|$ $= (16)^n$, then n is equal to

(1) 8 **(2)** 9 **(3)** 12 **(4)** 10

Q. 20. Let $I(x) = \int \frac{(x+1)}{x(1+xe^x)^2}dx, x > 0.$ $\lim_{x\to\infty} I(x) = 0$, then I(1) is equal to

(1) $\frac{e+1}{e+2} - \log_e(e+1)$ (2) $\frac{e+2}{e+1} + \log_e(e+1)$

(3) $\frac{e+2}{e+1} - \log_e(e+1)$ (4) $\frac{e+1}{e+2} + \log_e(e+1)$

Section B

Q. 21. Let A = {0, 3, 4, 6, 7, 8, 9, 10} and R be the relation defined on A such that R =$(x, y) \in$ A × A: $x - y$ is odd positive integer or $x - y = 2$}. The minimum number of elements that must be added to the relation R, so that it is a symmetric relation, is equal to ______.

Q. 22. Let $[t]$ denote the greatest integer $\le t$, If the constant term in the expansion of $\left(3x^2 - \frac{1}{2x^5}\right)^7$ is α, then $[\alpha]$ is equal to _____.

Q. 23. Let λ_1, λ_2 be the values of λ for which the points $\left(\frac{5}{2}, 1, \lambda\right)$ and (–2, 0, 1) are at equal distance from the plane $2x + 3y - 6z + 7 = 0$. If $\lambda_1 > \lambda_2$, then the distance of the point $(\lambda_1 - \lambda_2, \lambda_2, \lambda_1)$ from the line $\frac{x-5}{1} = \frac{y-1}{2} = \frac{z+7}{2}$ is

Q. 24. If the solution curve of the differential equation $(y - 2\log_e x)dx + (x \log_e x^2)\, dy = 0$, $x > 1$ passes through the points $\left(e, \frac{4}{3}\right)$ and (e^4, α), then α is equal to ______.

Q. 25. Let $\vec{a} = 6\hat{i} + 9\hat{j} + 12\hat{k}$, $\vec{b} = \alpha\hat{i} + 11\hat{j} - 2\hat{k}$ and $\vec{c}$ be vectors such that $\vec{a} \times \vec{c} = \vec{a} \times \vec{b}$. If $\vec{a}.\vec{c} = -12$, $\vec{c}.(\hat{i} - 2\hat{j} + \hat{k}) = 5$, then $\vec{c}(\hat{i} + \hat{j} + \hat{k})$ is equal to ________.

Q. 26. The largest natural number n such that $3n$ divides 66! is ______.

Q. 27. If a_0 is the greatest term in the sequence $a_n = \frac{n^3}{n^4 + 147}, n = 1, 2, 3, ...,$ then a is equal to ________________.

Q. 28. Let the mean and variance of 8 numbers x, y, 10, 12, 6, 12, 4, 8 be 9 and 9.25 respectively. If $x > y$, then $3x - 2y$ is equal to _________.

Q. 29. Consider a circle $C_1 : x^2 + y^2 - 4x - 2y = \alpha - 5$. Let its mirror image in the line $y = 2x + 1$ be another circle $C_2 : 5x^2 + 5y^2 - 10fx - 10gy + 36 = 0$. Let r be the radius of C_2. Then $\alpha + r$ is equal to _________.

Q. 30. Let $[t]$ denote the greatest integer $\le t$. Then $\frac{2}{\pi}\int_{\pi/6}^{5\pi/6} (8\,[\text{cosec}\ x] - 5\,[\cot x])\, dx$ is equal to _______.

Answer Key

Q. No.	Answer	Topic Name	Chapter Name
1	(3)	Area between the curves	Integral Calculus
2	(3)	Algebra of matrices	Matrices
3	(4)	Negation of a statement	Mathematical Reasoning
4	(4)	Circumcentre	Straight line
5	(3)	Cube root of unity	Cubic Equation
6	(4)	*r* things out of *n* things	Permutation and Combination
7	(2)	Coefficient of a term	Binomial theorem
8	(1)	Parabola	Conic Section
9	(1)	Sum of *n* terms	Sequences and series
10	(2)	Shortest distance	Three dimensional geometry
11	(1)	Number of ways	Permutation and Combination
12	(2)	Collinearity	Vector algebra
13	(1)	Conditional probability	Probability
14	(2)	Roots of equation	Complex numbers
15	(3)	Limits of trigonometry	Limits
16	(4)	Number of ways	Permutation and Combination
17	(2)	Higher order derivatives	Differentiability
18	(1)	Equation of plane	Three dimensional geometry
19	(4)	Adjoint	Matrices and Determinants
20	(3)	Indefinite Integral	Integral Calculus
21	[19]	Symmetric relation	Relation and Function
22	[1275]	General term	Binomial theorem
23	[9]	Plane	Three dimensional geometry

24	[3]	Linear Differential Equation	Differential equation
25	[11]	Algebra of vectors	Vector algebra
26	[31]	Remainder theorem	Binomial theorem
27	[5]	Maxima/Minima	Application of derivatives
28	[25]	Mean, Variance	Statistics
29	[2]	Circle	Conic Section
30	[14]	Definite Integral	Integral Calculus

Solutions

Section A

1. Option (3) is correct.

The given curves are

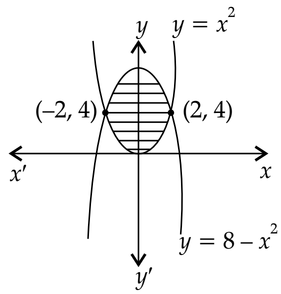
$x^2 \le y, y \le 8 - x^2; y \le 7$

On solving, we get

$x^2 = 8 - x^2$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$



So, area $= 2\left[\int_0^4 \sqrt{y}\,dy + \int_4^7 \sqrt{8-y}\,dy\right]$

$= 2\left\{\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 + \left[\frac{-(8-y)^{\frac{3}{2}}}{\frac{3}{2}}\right]_4^7\right\}$

$= \frac{4}{3}\{8 - 1 + 8\} = \frac{4}{3} \times 15 = 20$ sq. units

2. Option (3) is correct.

Here, P $= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, A $= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Here, $PP^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

$= \begin{bmatrix} \frac{3}{4} + \frac{1}{4} & -\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \\ \frac{-\sqrt{3}}{4} + \frac{\sqrt{3}}{4} & \frac{1}{4} + \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

| | $P^TP = I$

$\because Q = PAP^T$

$\Rightarrow Q^{2007} = (PAP^T)(PAP^T)$......... 2007 time

$= PA^{2007}P^T$

As, A $= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$A^{2007} = \begin{bmatrix} 1 & 2007 \\ 0 & 1 \end{bmatrix}$

Hence, $P^TQ^{2007}P = A^{2007} = \begin{bmatrix} 1 & 2007 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2007 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow a = 1, b = 2007, c = 0, d = 1$

$\therefore 2a + b - 3c - 4d = 2(1) + 2007 - 3(0) - 4(1)$

$= 2 + 2007 - 4 = 2005$

HINT:

Transpose the given matrix and multiply the matrices to solve further.

3. Option (4) is correct.

Given: $(p \to q) \to (q \to p)$

Negation of above statement is

$\sim[(p \to q) \to (q \to p)]$

$\equiv \sim[\sim p \to q \wedge q \to p]$

$\equiv p \to q \wedge \sim q \to p$

$\equiv \sim p \vee q \wedge q \wedge \sim p]$

$\equiv q \wedge (\sim p)$

4. Option (4) is correct.

We have,

$4x + 3y = 69$ ...(i)

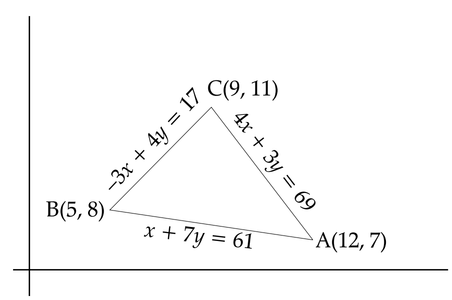
$4y - 3x = 17$ ...(ii)

$x + 7y = 61$ ...(iii)

On solving (i) and (iii), we get

$x = 12$, and $y = 7$

So, A ≡ (12, 7)



On solving (ii) and (iii), we get

$x = 5$ and $y = 8$

So, B ≡ (5, 8)

Hence, circumcentre $\equiv \left(\frac{12+5}{2}, \frac{7+8}{2}\right)$

$\equiv \left(\frac{17}{2}, \frac{15}{2}\right)$

$\therefore \alpha = \frac{17}{2}, \beta = \frac{15}{2}$

$\therefore (\alpha - \beta)^2 + (\alpha + \beta) = \left(\frac{17}{2} - \frac{15}{2}\right)^2 + \left(\frac{17}{2} + \frac{15}{2}\right)$

$= (1)^2 + (16) = 17$

HINT:
Circumcentre of a right triangle is the midpoint of hypotenuse of the triangle.

5. Option (3) is correct.

Given cubic equation is :

$x^3 + bx + c = 0$

$\because \alpha, \beta, \gamma$ are the roots of above equation.

And $\beta\gamma = 1 = -\alpha$

So, product of roots $= -c$

$\Rightarrow \alpha\beta\gamma = -c \Rightarrow c = 1$

Since, $\alpha = -1$ is the root. So,

$\Rightarrow -1 - b + c = 0 \Rightarrow 1 - b = 1 \Rightarrow b = 0$

The given equation becomes $x^3 + 1 = 0$

So, roots are $-1, -\omega, -\omega^2$

$\therefore b^3 + 2c^3 - 3\alpha^3 - 6\beta^3 - 8\gamma^3$

$= 0 + 2 - 3(-1)^3 - 6(-\omega)^3 - 8(-\omega^2)^3$

$= 2 + 3 + 6\omega^3 + 8\omega^6$

$= 5 + 6 + 8 = 19$

6. Option (4) is correct.

Since, $n(A) = 5, n(B) = 2$

$\Rightarrow n(A \times B) = n(A) \times n(B)$

$= 5 \times 2 = 10$

So, number of subsets having 3 elements $= {}^{10}C_3$

Number of subsets having 4 elements $= {}^{10}C_4$

Number of subsets having 5 elements $= {}^{10}C_5$

Number of subsets having 6 elements $= {}^{10}C_6$

$\therefore$ No. of subsets $= {}^{10}C_3 + {}^{10}C_4 + {}^{10}C_5 + {}^{10}C_6$

$= 120 + 210 + 252 + 210 = 792$

HINT:
No of subsets having r elements out of total n elements $= {}^nC_r$

7. Option (2) is correct.

Given: ${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1}$

$= 1 : 5 : 20$

$\Rightarrow \frac{n!}{(r-1)!(n-r+1)!} \times \frac{r!(n-r)!}{n!} = \frac{1}{5}$

$\Rightarrow \frac{r}{(n-r+1)} = \frac{1}{5}$

$\Rightarrow 5r = n - r + 1$

$\Rightarrow n = 6r - 1$ ...(i)

Also, $\frac{n}{r!(n-r)!} \times \frac{(r+1)!(n-r-1)!}{n!} = \frac{5}{20} = \frac{1}{20}$

$\Rightarrow \frac{(r+1)}{(n-r)} = \frac{1}{4}$

$\Rightarrow 4r + 4 = n - r$

$\Rightarrow n = 5r + 4$ ...(ii)

From (i) and (ii), we get

$6r - 1 = 5r + 4$

$\Rightarrow r = 5$

So, $n = 5(5) + 4 = 29$

So, coefficient of 4th terms $= {}^nC_3 = {}^{29}C_3$

$= \frac{29!}{3!26!} = \frac{29 \times 28 \times 27}{3 \times 2} = 3654$

HINT:
In the expansion of $(a + b)^n$, the general term is $T_{r+1} = {}^nC_r (a)^{n-r} b^r$

8. Option (1) is correct.

$y^2 = 20x, y = mx + c$

Put value of x

$y^2 = 20\left(\frac{y-c}{m}\right)$

$\Rightarrow y^2 - \frac{20}{m}y + \frac{20}{m}c = 0$ ...(i)

Since, centroid = (10, 10)

So, $\frac{y_1 + y_2 + 0}{3} = 10$

$\Rightarrow y_1 + y_2 = 30$

From (1),

Sum of roots $= \frac{20}{m} = 30 \Rightarrow m = \frac{2}{3}$

Also, $c - m = 6 \Rightarrow c = 6 + \frac{2}{3} = \frac{20}{3}$

Now, the equation is :

$y^2 - \frac{20}{2} \times 3y + \frac{20}{2} \times 3 \times \frac{20}{3} = 0$

$\Rightarrow (y - 20)(y - 10) = 0$

$\Rightarrow y = 10, 20 \Rightarrow x = 5, x = 20$

$\therefore P \equiv (5, 10), Q \equiv (20, 20)$

So, $(PQ)^2 = (20 - 5)^2 + (20 - 10)^2$

$= 225 + 100 = 325$

9. Option (1) is correct.

$\because S_k = \frac{1 + 2 + ... + k}{k}$

$= \frac{k(k+1)}{2k} = \frac{k+1}{2}$

$\Rightarrow S_k^2 = \left(\frac{k+1}{2}\right)^2 = \frac{k^2 + 1 + 2k}{4}$

$\Rightarrow \sum_{j=1}^{n} S_j^2 = \frac{1}{4}\left[\sum_{j=1}^{n} k^2 + \sum_{j=1}^{n} 1 + 2\sum_{j=1}^{n} k\right]$

$= \frac{n}{4}\left[\frac{(n+1)(2n+1)}{6} + 1 + n + 1\right]$

$= \frac{n}{24}[2n^2 + 9n + 13]$

On comparing, we get

A = 24, B = 2, C = 9, D = 13

(1) A + B = 24 + 2 = 26, divisible by 13

(2) A + B = 26

5 (D – C) = 5 (13 – 9) = 20

$\therefore$ 26 ≠ 20

(3) A + C + D = 46, which is divisible by 2

(4) A + B + D = 39, which is not divisible by 5.

10. Option (2) is correct.

The given lines are :

$$\frac{x-4}{4}=\frac{y+2}{5}=\frac{z+3}{3} \text{ and } \frac{x-1}{3}=\frac{y-3}{4}=\frac{z-4}{2}$$

So, $\vec{b}_1 = 4\hat{i}+5\hat{j}+3\hat{k}$

$\vec{b}_2 = 3\hat{i}+4\hat{j}+2\hat{k}$

$\vec{a}_1 = 4\hat{i}-2\hat{j}-3\hat{k}, \vec{a}_2 = \hat{i}+3\hat{j}+4\hat{k}$

$$\therefore \vec{b}_1\times\vec{b}_2 = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 5 & 3\\ 3 & 4 & 2\end{vmatrix}$$

$= -2\hat{i}+\hat{j}+\hat{k}$

Shortest distance, $d = \left|\frac{(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$

$$= \left|\frac{(3\hat{i}-5\hat{j}-7\hat{k}).(-2\hat{i}+\hat{j}+\hat{k})}{\sqrt{4+1+1}}\right|$$

$$= \left|\frac{-6-5-7}{\sqrt{6}}\right| = \frac{18}{\sqrt{6}} = 3\sqrt{6} \text{ units}$$

HINT:

Shortest distance between two lines is:

$$d = \left|\frac{(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$$

11. Option (1) is correct.

In the given word,

vowels are : I, E, E, E, E

Consonants are : N, D, P, N, D, N, C

So, number of words $= \frac{8!}{3!2!}\times\frac{5!}{4!}$

$$= \frac{8\times7\times6\times5\times4}{2}\times5 = 16800$$

12. Option (2) is correct.

Given: Points with position vectors

$\alpha\hat{i}+10\hat{j}+13\hat{k}, 6\hat{i}+11\hat{j}+11\hat{k}$

and $\frac{9}{2}\hat{i}+\beta\hat{j}-8\hat{k}$ are collinear.

$$\text{So, } \frac{\alpha-6}{6-\frac{9}{2}} = \frac{10-11}{11-\beta} = \frac{13-11}{11+8}$$

$$\Rightarrow \frac{2(\alpha-6)}{3} = \frac{-1}{11-\beta} = \frac{2}{19}$$

$$\Rightarrow \frac{2}{3}(\alpha-6) = \frac{2}{19}$$

$\Rightarrow 19\alpha - 114 = 3 \Rightarrow 19\alpha = 117$

$$\Rightarrow \alpha = \frac{117}{19}$$

And, $\frac{-1}{11-\beta} = \frac{2}{19}$

$\Rightarrow -19 = 22 - 2\beta$

$\Rightarrow 2\beta = 41$

$$\Rightarrow \beta = \frac{41}{2}$$

$$\therefore (19\alpha-6\beta)^2 = \left(19\times\frac{117}{19}-\frac{6\times41}{2}\right)^2$$

$= (117 - 123)^2 = 36$

HINT:

If point $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_1)$ are collinear, then $\frac{\alpha_1-\alpha_2}{\alpha_2-\alpha_3} = \frac{\beta_1-\beta_2}{\beta_2-\beta_3} = \frac{\gamma_1-\gamma_2}{\gamma_2-\gamma_3}$.

13. Option (1) is correct.

Given: $P(A) = \frac{20}{100} = \frac{2}{10}$

$P(B) = \frac{30}{100} = \frac{3}{10}$; $P(C) = \frac{50}{100} = \frac{5}{10}$

Let E → Event that the bolt is defective.

So, $P(E/A) = \frac{3}{100}, P\left(\frac{E}{B}\right) = \frac{4}{100}, P\left(\frac{E}{C}\right) = \frac{2}{100}$

So, P(C/E)

$$= \frac{P\left(\frac{E}{C}\right)\times P(C)}{P\left(\frac{E}{A}\right)\times P(A)+P\left(\frac{E}{B}\right)\times P(B)+P\left(\frac{E}{C}\right)\times P(C)}$$

$$= \frac{\frac{5}{10}\times\frac{2}{100}}{\frac{3}{100}\times\frac{2}{10}+\frac{4}{100}\times\frac{3}{10}+\frac{2}{100}\times\frac{5}{10}}$$

$$= \frac{10}{6+12+10} = \frac{10}{28} = \frac{5}{14}$$

14. Option (2) is correct.

Given: $|z + 2| = z + 4(1 + i)$

Also, $z = \alpha + i\beta$

$\therefore |z + 2| = |\alpha + i\beta + 2| = (\alpha + i\beta) + 4 + 4i$

$\Rightarrow |(\alpha + 2) + i\beta| = (\alpha + 4) + i(\beta + 4)$

$$\Rightarrow \sqrt{(\alpha+2)^2+\beta^2} = (\alpha+4)+i(\beta+4)$$

$\Rightarrow \beta + 4 = 0 \Rightarrow \beta = -4$

Now, $(\alpha + 2)^2 + \beta^2 = (\alpha + 4)^2$

$\Rightarrow \alpha^2 + 4 + 4\alpha + \beta^2 = \alpha^2 + 16 + 8\alpha$

$\Rightarrow 4 + 4\alpha + 16 = 16 + 8\alpha$

$\Rightarrow 4\alpha = 4 \Rightarrow \alpha = 1$

So, $\alpha + \beta = -3$ and $\alpha\beta = -4$

$\therefore$ Required equation is

$x^2 - (-3-4)\,x + (-3)(-4) = 0$

$\Rightarrow x^2 + 7x + 12 = 0$

15. Option (3) is correct.

$$\lim_{x\to 0}\left[\left(\frac{1-\cos^2(3x)}{\cos^3(4x)}\right)\left(\frac{\sin^3(4x)}{\left(\log_e(2x+1)\right)^5}\right)\right]$$

$$= \lim_{x\to 0}\left[\frac{1-\cos^2(3x)}{9x^2}\times\frac{9x^2}{\cos^3(4x)}\right]\times\frac{\dfrac{\sin^3 4x}{(4x)^3}\times 64x^3}{\left[\dfrac{\log_e(2x+1)}{2x}\right]^5\times(2x)^5}$$

$$=\left[\frac{1\times 9\times 1}{(1)}\right]\times\left[\frac{1\times 64}{1\times 32}\right]$$

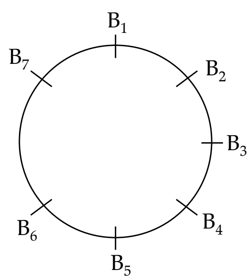
$= 9 \times 2 = 18$

16. Option (4) is correct.

We have,

Number of girls = 5

Number of boys = 7



So, number of ways of arranging boys around the table = 6! and 5 girls can be arranged in 7 gaps in $^7P_5$ ways

$\therefore$ Required no. of ways = $6! \times {}^7P_5$

$= 126 \times (5!)^2$

17. Option (2) is correct.

$$f(x) = \frac{\sin x + \cos x - \sqrt{2}}{\sin x - \cos x}$$

$$= \frac{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x - 1}{\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x}$$

$$= \frac{\cos\left(x-\frac{\pi}{4}\right)-1}{\sin\left(x-\frac{\pi}{4}\right)}$$

$$= \frac{-2\sin^2\left(\frac{x}{2}-\frac{\pi}{8}\right)}{2\sin\left(\frac{x}{2}-\frac{\pi}{8}\right)\cos\left(\frac{x}{2}-\frac{\pi}{8}\right)}$$

$$\Rightarrow f(x) = -\tan\left(\frac{x}{2}-\frac{\pi}{8}\right)$$

$$\Rightarrow f'(x) = -\frac{1}{2}\sec^2\left(\frac{x}{2}-\frac{\pi}{8}\right)$$

$$\Rightarrow f''(x) = -\frac{1}{2}.2\sec\left(\frac{x}{2}-\frac{\pi}{8}\right).\sec\left(\frac{x}{2}-\frac{\pi}{8}\right)\tan\left(\frac{x}{2}-\frac{\pi}{8}\right)\times\frac{1}{2}$$

$$= -\frac{1}{2}\sec^2\left(\frac{x}{2}-\frac{\pi}{8}\right).\tan\left(\frac{x}{2}-\frac{\pi}{8}\right)$$

Now, $f\left(\frac{7\pi}{12}\right)f''\left(\frac{7\pi}{12}\right)$

$$= -\tan\left(\frac{7\pi}{24}-\frac{\pi}{8}\right)\times\frac{-1}{2}\sec^2\left(\frac{7\pi}{24}-\frac{\pi}{8}\right)\times\tan\left(\frac{7\pi}{24}-\frac{\pi}{8}\right)$$

$$= \frac{1}{2}\tan^2\left(\frac{\pi}{6}\right)\times\sec^2\frac{\pi}{6}$$

$$= \frac{1}{2}\times\frac{1}{3}\times\frac{4}{3} = \frac{2}{9}$$

18. Option (1) is correct.

Equation of plane P containing the given lines is

$(x + 2y + 3z - 4) + \lambda(2x + y - z + 5) = 0$

$\Rightarrow (1 + 2\lambda)\,x + (2 + \lambda)y + (3 - \lambda)\,z + (-4 + 5\lambda) = 0$

Now, plane P is perpendicular to plane P'

$$\vec{r} = (\hat{i}-\hat{j}) + \lambda(\hat{i}+\hat{j}+\hat{k}) + \mu(\hat{i}-2\hat{j}+3\hat{k})$$

So, normal to plane P' is

$$\vec{n} = (\hat{i}+\hat{j}+\hat{k})\times(\hat{i}-2\hat{j}+3\hat{k})$$

$$\Rightarrow \vec{n} = 5\hat{i}-2\hat{j}-3\hat{k}$$

$\therefore$ P and P' are perpendicular

$\therefore 5(1 + 2\lambda) - 2(2 + \lambda) - 3(3 - \lambda) = 0$

$\Rightarrow 5 + 10\lambda - 4 - 2\lambda - 9 + 3\lambda = 0$

$\Rightarrow 11\lambda = 8 \Rightarrow \lambda = \frac{8}{11}$

$$\therefore \text{P}: \left(1+\frac{16}{11}\right)x + \left(2+\frac{8}{11}\right)y + \left(3-\frac{8}{11}\right)z + \left(5\times\frac{8}{11}-4\right) = 0$$

i.e., $27x + 30y + 25z = 4$

which is same as $ax + by + cz = 4$

$\therefore a = 27, b = 30$ and $c = 25$

$\Rightarrow a - b + c = 27 - 30 + 25 = 22$

HINT:

When two planes are perpendicular, then dot product of their normals is zero.

19. Option (4) is correct.

We have,

$$|\text{A}| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 2(4-1) - 1(2-0) + 0$$

$= 6 - 2 = 4$

So, $|2\text{A}| = 2^3\,|\text{A}| = 8 \times 4 = 32$

Now, $|\text{adj (adj (adj 2A))}| = |2\text{A}|^{(n-1)^3}$

$= (32)^{2^3} = 32^8$

$\Rightarrow 16^n = (32)^8 = 2^8 \times 16^8$

$\Rightarrow 16^n = 16^{2+8} \Rightarrow n = 10$

20. Option (3) is correct.

$$\text{I} = \int \frac{x+1}{x(1+xe^x)^2}\,dx$$

Put $1 + xe^x = t \Rightarrow xe^x = t - 1$

$\Rightarrow (xe^x + e^x)\, dx = dt$

$\Rightarrow e^x (x + 1)\, dx = dt$

$$\therefore I = \int \frac{dt}{e^x . x t^2} = \int \frac{dt}{(t-1)t^2}$$

$$\text{Let } \frac{1}{t^2(t-1)} = \frac{A}{(t-1)} + \frac{Bt + C}{t^2}$$

$\Rightarrow 1 = At^2 + (Bt + C)(t - 1)$

Comparing coefficients of t^2, t and constant terms, we get

$A + B = 0, C - B = 0, -C = 1$

On solving above equations, we get

$C = -1, =B, A = 1$

$$\therefore \; I = \int \frac{1}{t-1} dt + \int \frac{-t-1}{t^2} dt$$

$$= \int \frac{1}{t-1} dt - \int \frac{1}{t} dt - \int \frac{1}{t^2} dt$$

$$= \log|t-1| - \log|t| + \frac{1}{t} + C$$

$$\Rightarrow I = \log|xe^x| - \log|1 + xe^x| + \frac{1}{1 + xe^x} + c$$

$$= \log\left|\frac{xe^x}{1 + xe^x}\right| + \frac{1}{1 + xe^x} + C$$

Now, $\lim_{x \to \infty} I(x) = 0$

$$\Rightarrow \lim_{x \to \infty} \left\{ \log\left|\frac{xe^x}{1 + xe^x}\right| + \frac{1}{1 + xe^x} + C \right\} = 0$$

$$\Rightarrow \lim_{x \to \infty} \left\{ \log\left(\frac{e^x}{\frac{1}{x} + e^x} \right) + \frac{\frac{1}{x}}{\frac{1}{x} + e^x} + C \right\}$$

$\Rightarrow 0 + 0 + C = 0 \Rightarrow C = 0$

$$\therefore I(x) = \log\left|\frac{xe^x}{1 + xe^x}\right| + \frac{1}{1 + xe^x}$$

$$\Rightarrow I(1) = \log\left|\frac{e}{1+e}\right| + \frac{1}{1+e} = 1 - \log(1+e) + \frac{1}{1+e}$$

$$= \frac{2+e}{1+e} - \log|1+e|$$

Section B

21. Correct answer is [19].

We have, $A = \{0, 3, 4, 6, 7, 8, 9, 10\}$

Case I: $x - y$ is odd, if one is odd and one is even and $x > y$.

$\therefore$ Possibilites are {(3, 0), (4, 3), (6, 3), (7, 6), (7, 4), (7, 0), (8, 7), (8, 3), (9, 8), (9, 6), (9, 4), (9, 0), (10, 9), (10, 7), (10, 3)}

No. of cases = 15

Case II: $x - y = 2$

$\therefore$ Possibilities are {(6, 4), (8, 6), (9, 7), (10, 8)}

$\therefore$ No. of cases = 4

So, minimum ordered pair to be added = 15 + 4 = 19

HINT:

Any relations said to be symmetric if $(a, b) \in R$ $\Rightarrow (b, a) \in R$

22. Correct answer is [1275].

Let T_{r+1} be the constant term.

$$T_{r+1} = {}^7C_r (3x^2)^{7-r} \left(\frac{-1}{2x^5}\right)^r$$

For constant term, power of x should be zero.

i.e., $14 - 2r - 5r = 0$

$\Rightarrow 14 = 7r \Rightarrow r = 2$

Now, constant term = α

$$\Rightarrow {}^7C_2 (3)^5 \left(\frac{-1}{2}\right)^2 = \alpha$$

$$\Rightarrow 21 \times 243 \times \frac{1}{4} = \alpha$$

$\Rightarrow [\alpha] = [1275.75] = 1275$

HINT:

Let $(a + b)^n$, then $T_{r+1} = {}^nC_r\, a^{n-r} . b^r$

23. Correct answer is [9].

Since $\left(\frac{5}{2}, 1, \lambda\right)$ and $(-2, 0, 1)$ are equidistant from plane $2x + 3y - 6z + 7 = 0$

$$\therefore \left| \frac{2\left(\frac{5}{2}\right) + 3(1) - 6(\lambda) + 7}{\sqrt{2^2 + 3^2 + 6^2}} \right| = \left| \frac{2(-2) + 3(0) - 6(1) + 7}{\sqrt{2^2 + 3^2 + 6^2}} \right|$$

$\Rightarrow |5 + 3 - 6\lambda + 7| = |-4 - 6 + 7|$

$\Rightarrow 15 - 6\lambda = 3$ or $15 - 6\lambda = -3$

$\Rightarrow 6\lambda = 12$ or $6\lambda = 18$

$\Rightarrow \lambda = 2$ or $\lambda = 3$

$\because \lambda_1 > \lambda_2$

$\therefore \lambda_1 = 3$ and $\lambda_2 = 2$

So, point will be (1, 2, 3)

Let $M_0 = (1, 2, 3)$

M_1 is the point through which line passes i.e, (5, 1, –7)

and $\vec{s} = \hat{i} + 2\hat{j} + 2\hat{k}$

$$\therefore \overrightarrow{M_0M_1} = 4\hat{i} - \hat{j} - 10\hat{k}$$

$$\text{Now, required distance} = \left| \frac{\overrightarrow{M_0M_1} \times \vec{s}}{|\vec{s}|} \right|$$

$$= \frac{|(4\hat{i} - \hat{j} - 10\hat{k}) \times (\hat{i} + 2\hat{j} + 2\hat{k})|}{\sqrt{1 + 4 + 4}}$$

$$= \frac{|18\hat{i} - 18\hat{j} + 9\hat{k}|}{3} = 9$$

24. Correct answer is [3].

The given differential equation is,

$(y - 2\log x)\, dx + (x \log x^2)\, dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{(2\log x - y)}{2x \log x}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{2x \log x} = \frac{1}{x}$$

It is a linear differential equation.

$$\therefore \text{I.F.} = e^{\int \frac{1}{2x \log x} dx}$$

Put $\log x = t \Rightarrow \frac{1}{x}dx = dt$

$\therefore$ I.F. $= e^{\int \frac{1}{2t}dt} = e^{\log(t)^{\frac{1}{2}}} = \sqrt{t} = \sqrt{\log x}$

So, required solution is,

$$y\sqrt{\log x} = \int \frac{\sqrt{\log x}}{x}dx$$

$$\log x = v \Rightarrow \frac{1}{x}dx = dv$$

$$\Rightarrow y\sqrt{\log x} = \int \sqrt{v}\, dv + C$$

$$\Rightarrow y\sqrt{\log x} = \frac{2v^{3/2}}{3} + C$$

$$\Rightarrow y\sqrt{\log x} = \frac{2}{3}(\log x)^{3/2} + C$$

Now, this curve passes through $\left(e, \frac{4}{3}\right)$ and (e^4, α)

$$\therefore \frac{4}{3}\sqrt{\log e} = \frac{2}{3}(\log e)^{3/2} + C$$

$$\Rightarrow C = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$$

Also, $\alpha\sqrt{\log e^4} = \frac{2}{3}(\log e^4)^{3/2} + \frac{2}{3}$

$$\Rightarrow 2\alpha = \frac{2}{3} \times (4)^{3/2} + \frac{2}{3} = \frac{16}{3} + \frac{2}{3} = \frac{18}{3}$$

$\Rightarrow \alpha = 3$

HINT:

Reduce the given differential equation to linear differential equation and find its solution.

25. Correct answer is [11].

Let $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Now, $\vec{a}.\vec{c} = -12$

$\Rightarrow 6C_1 + 9C_2 + 12C_3 = -12$ ...(i)

Also, $\vec{c}.(\hat{i} - 2\hat{j} + \hat{k}) = 5$

$\Rightarrow C_1 - 2C_2 + C_3 = 5$ ...(ii)

Now, $\vec{a} \times \vec{c} = \vec{a} \times \vec{b}$

$\Rightarrow \vec{a} \times (\vec{c} - \vec{b}) = 0$

$\Rightarrow \vec{a}$ is parallel to $(\vec{c} - \vec{b})$

$\Rightarrow \vec{a} = \lambda(\vec{c} - \vec{b})$

$$\Rightarrow 6\hat{i} + 9\hat{j} + 12\hat{k} = \lambda(c_1 - \alpha)\hat{i} + \lambda(c_2 - 11)\hat{j} + \lambda(c_3 + 2)\hat{k}$$

On comparing, we get

$$c_1 = \frac{6}{\lambda} + \alpha, c_2 = \frac{9}{\lambda} + 11, c_3 = \frac{12}{\lambda} - 2$$

Put there values in (ii), we get

$$\frac{6}{\lambda} + \alpha - \frac{18}{\lambda} - 22 + \frac{12}{\lambda} - 2 = 5$$

$\Rightarrow \alpha = 29$

From (i) and values of C_1, C_2, C_3, and α we have

$$6\left(\frac{6}{\lambda} + 29\right) + 9\left(\frac{9}{\lambda} + 11\right) + 12\left(\frac{12}{\lambda} - 2\right) = -12$$

$$\Rightarrow \frac{261}{\lambda} = -261 \Rightarrow \lambda = -1$$

So, $C_1 = 23$, $C_2 = 2$, $C_3 = -14$

$$\therefore \vec{c}.(\hat{i} + \hat{j} + \hat{k}) = (23\hat{i} + 2\hat{j} + -14\hat{k}).(\hat{i} + \hat{j} + \hat{k})$$

$= 23 + 2 - 14 = 11$

HINT:

$$\vec{a} \times \vec{c} = \vec{a} \times \vec{b} \Rightarrow \vec{a} \,||\, (\vec{c} - \vec{b}) \Rightarrow a = \lambda(\vec{c} - \vec{b})$$

26. Correct answer is [31].

We have,

$$\left[\frac{66}{3}\right] = 22, \left[\frac{66}{3^2}\right] = 7, \left[\frac{66}{3^3}\right] = 2$$

Highest powers of 3 is greater than 66. So, their g.i.f. is always 0

$\therefore$ Required natural number = 22 + 7 + 2 = 31

27. Correct answer is [5].

Let $y = \frac{x^3}{x^4 + 147}$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^4 + 147) \times 3x^2 - x^3(4x^3)}{(x^4 + 147)^2}$$

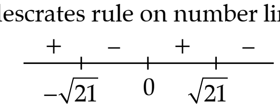
$$= \frac{3x^6 + 441x^2 - 4x^6}{(x^4 + 147)^2} = \frac{441x^2 - x^6}{(x^4 + 147)^2}$$

For maxima/minima, put $\frac{dy}{dx} = 0$

$\Rightarrow 441\, x^2 - x^6 = 0 \Rightarrow x^4 = 441$

$\Rightarrow x = \pm\sqrt{21}, \pm\sqrt{21}i$

Now, by descrates rule on number line we have



Since sign changes from negative to positive at 0.

$\therefore$ Maximum value of is at $x = \sqrt{21} = 4.58$

Now, 4 < 4.5 < 5

$$\therefore y \text{ at } x = 4 = \frac{64}{403} = 0.159$$

$$y \text{ at } x = 5 = \frac{125}{772} = 0.162$$

So, y is maximum at $x = 5$

$\therefore \alpha = 5$

HINT:

For maximum value, find $\frac{dy}{dx}$ and then observe the change in signs using decrates rule.

28. Correct answer is [25].

x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
x	$x - 9$	$(x - 9)^2$
y	$y - 9$	$(y - 9)^2$
10	1	1
12	3	9
6	–3	9
12	3	9
4	–5	25
8	–1	1
$x+y+92$		$(x-9)^2 + (y-9)^2 + 54$

Now, mean $(\bar{x}) = 9$

$$\Rightarrow \frac{x+y+52}{8} = 9$$

$\Rightarrow x + y = 20$ ...(i)

Also, variance = 9.25

$$\Rightarrow \frac{(x-9)^2 + (y-9)^2 + 54}{8} = 9.25$$

$\Rightarrow x^2 + y^2 + 81 + 81 - 2 \times 9(x + y) = 20$

$\Rightarrow x^2 + y^2 - 18 \times 20 = -142$

$\Rightarrow x^2 + y^2 = 218$

$\Rightarrow x^2 + (20 - x)^2 = 218$

$\Rightarrow x^2 + 400 + x^2 - 40x = 218$

$\Rightarrow 2x^2 - 40x + 182 = 0$

$$\Rightarrow x = \frac{40 \pm 12}{4}$$

$\Rightarrow x = 13$ or $x = 7 \Rightarrow y = 7$ or $y = 13$

But $x > y$

$\therefore x = 13$ and $y = 7$

So, $3x - 2y = 39 - 14 = 25$

29. Correct answer is [2].

We have,

$C_1 : x^2 + y^2 - 4x - 2y = \alpha - 5$

$C_1 : (x - 2)^2 + (y - 1)^5 - 5 = \alpha - 5$

$C_1 : (x - 2)^2 + (y - 1)^2 = (\sqrt{\alpha})^2$

So, centre and radius of C_1 are (2, 1) and $\sqrt{\alpha}$ respectively

Now, image of (2, 1) along the line $y = 2x + 1$ is,

$$\frac{x-2}{2} = \frac{y-1}{-1} = \frac{-2(4-1+1)}{2^2 + (-1)^2}$$

$$\Rightarrow \frac{x-2}{2} = \frac{y-1}{-1} = \frac{-8}{5}$$

$$\Rightarrow x = \frac{-6}{5} \text{ and } y = \frac{13}{5}$$

Now, $\left(\frac{-6}{5}, \frac{13}{5}\right)$ will be the centre of C_2

$$\therefore f = \frac{6}{5} \text{ and } g = \frac{-13}{5}$$

Now, radius of $C_2 = r = \sqrt{f^2 + g^2 - \frac{36}{5}}$

$$\Rightarrow r = \sqrt{\frac{36}{25} + \frac{169}{25} - \frac{36}{5}} = 1$$

$\because r = 1$ so, $\alpha = 1$

$\therefore \alpha + r = 1 + 1 = 2$

HINT:

Image of a point (x_1, y_1) w.r.t. $ax + by + c = 0$ is (x, y), then

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{(a^2 + b^2)}$$

30. Correct answer is [14].

$$\text{Let I} = \frac{2}{\pi} \int_{\frac{\pi}{6}}^{5\frac{\pi}{6}} \{8[\text{cosec } x] - 5[\cot x]\} dx$$

$$= \frac{2}{\pi}\left[8 \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [\text{cosec } x] dx - 5 \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [\cot x] dx\right]$$

$$= \frac{2}{\pi}\left[8 \int_{\pi/6}^{5\pi/6} dx - 5\left\{\int_{\pi/6}^{\pi/4} dx + \int_{\pi/4}^{\pi/2} 0.dx + \int_{\pi/2}^{3\pi/4} (-1) dx + \int_{3\pi/4}^{5\pi/6} (-2) dx\right\}\right]$$

$$= \frac{2}{\pi}\left[8 \times \left(\frac{5\pi}{6} \frac{-\pi}{6}\right) - 5\left\{\left(\frac{\pi}{4} - \frac{\pi}{6}\right) - \left(\frac{3\pi}{4} - \frac{\pi}{2}\right)\right\} - 2\left(\frac{5\pi}{6} - \frac{3\pi}{4}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{16\pi}{3} + \frac{5\pi}{3}\right] = 14$$

HINT:

Check the graph of [cosec x] and [cot x].

❑❑