JEE (Main) MATHEMATICS SOLVED PAPER

2023
25th Jan. Shift 1

Section A

Q. 1. The points of intersection of the line $ax + by = 0$, $(a \neq b)$ and the circle $x^2 + y^2 - 2x = 0$ are $A(\alpha, 0)$ and $B(1, \beta)$. The image of the circle with AB as a diameter in the line $x + y + 2 = 0$ is :

(1) $x^2 + y^2 + 3x + 3y + 4 = 0$
(2) $x^2 + y^2 + 3x + 5y + 8 = 0$
(3) $x^2 + y^2 - 5x - 5y + 12 = 0$
(4) $x^2 + y^2 + 5x + 5y + 12 = 0$

Q. 2. The distance of the point $(6 - 2\sqrt{2})$ from the common tangent $y = mx + c$, $m > 0$, of the curves $x = 2y^2$ and $x = 1 + y^2$ is:

(1) $\frac{14}{3}$ **(2)** $5\sqrt{3}$ **(3)** $\frac{1}{3}$ **(4)** 5

Q. 3. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non zero vectors such that $\vec{b}.\vec{c} = 0$ and $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{b} - \vec{c}}{2}$. If $\vec{d}$ be a vector such that $\vec{b}.\vec{d} = \vec{a}.\vec{b}$, then $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d})$ is equal to

(1) $-\frac{1}{4}$ **(2)** $\frac{1}{4}$ **(3)** $\frac{3}{4}$ **(4)** $\frac{1}{2}$

Q. 4. The vector $\vec{a} = -\hat{i} + 2\hat{j} + \hat{k}$ is rotated through a right angle, passing through the y-axis in its way and the resulting vector is $\vec{b}$. Then the projection of $3\vec{a} + \sqrt{2}\vec{b}$ on $\vec{c} = 5\hat{i} + 4\hat{j} + 3\hat{k}$ is:

(1) $2\sqrt{3}$ **(2)** 1 **(3)** $3\sqrt{2}$ **(4)** $\sqrt{6}$

Q. 5. Let $z_1 = 2 + 3i$ and $z_2 = 3 + 4i$. The set $S = \{z \in C: |z - z_1|^2 - |z - z_2|^2\}$ represents a

(1) hyperbola with the length of the transverse axis 7
(2) hyperbola with eccentricity 2
(3) straight line with the sum of its intercepts on the coordinate axes equals −18
(4) straight line with the sum of its intercepts on the coordinate axes equals 14

Q. 6. The mean and variance of the marks obtained by the students in a test are 10 and 4 respectively. Later, the marks of one of the students is increased from 8 to 12 . If the new mean of the marks is 10.2, then their new variance is equal to :

(1) 3.96 **(2)** 4.08 **(3)** 4.04 **(4)** 3.92

Q. 7. Let S_1 and S_2 be respectively the sets of all $a \in R - [0]$ for which the system of linear equations

$$ax + 2ay - 3az = 1$$
$$(2a + 1)x + (2a + 3)y + (a + 1)z = 2$$
$$(3a + 5)x + (a + 5)y + (a + 2)z = 3$$

has unique solution and infinitely many solutions. Then

(1) S_1 is an infinite set and $n(S_2) = 2$
(2) $S_1 = \Phi$ and $S_2 = R - \{0\}$
(3) $n(S_1) = 2$ and S_1 is an infinite set
(4) $S_1 = R - \{0\}$ and $S_2 = \Phi$

Q. 8. The value of

$$\lim_{n\to\infty} \frac{1 + 2 - 3 + 4 + 5 - 6 + ... + (3n-2) + (3n-1) - 3n}{\sqrt{2n^4 + 4n + 3} - \sqrt{n^4 + 5n + 4}}$$

(1) $\frac{3}{2}(\sqrt{2} + 1)$ **(2)** $\frac{3}{2\sqrt{2}}$

(3) $\frac{\sqrt{2} + 1}{2}$ **(4)** $3(\sqrt{2} + 1)$

Q. 9. The statement $(p \wedge (\sim q)) \Rightarrow (p \Rightarrow (\sim q))$ is

(1) a tautology
(2) a contradiction
(3) equivalent to $p \vee q$
(4) equivalent to $(\sim p) \vee (\sim q)$

Q. 10. Consider the lines L_1 and L_2 given by

$$L_1 : \frac{x-1}{2} = \frac{y-3}{1} = \frac{z-2}{2}$$

$$L_2 : \frac{x-2}{1} = \frac{y-2}{2} = \frac{z-3}{3}$$

A line L_3 having direction ratios 1, –1, –2, intersects L_1 and L_2 at the points P and Q respectively. Then the length of line segment PQ is

(1) $3\sqrt{2}$ **(2)** $4\sqrt{3}$ **(3)** 4 **(4)** $2\sqrt{6}$

Q. 11. Let $f(x) = \int \frac{2x}{(x^2 + 1)(x^2 + 3)} dx$.

If $f(3) = \frac{1}{2}(\log_e 5 - \log_e 6)$, then $f(4)$ is equal to

(1) $\log_e 19 - \log_e 20$
(2) $\log_e 17 - \log_e 18$
(3) $\frac{1}{2}(\log_e 19 - \log_e 17)$
(4) $\frac{1}{2}(\log_e 17 - \log_e 19)$

Q. 12. The minimum value of the function $f(x)=\int_0^2 e^{|x-t|}\,dt$ is:

(1) $e(e-1)$ **(2)** $2(e-1)$
(3) 2 **(4)** $2e-1$

Q. 13. Let M be the maximum value of the product of two positive integers when their sum is 66. Let the sample space S = $\left\{x \in Z : x(66-x) \ge \frac{5}{9}M\right\}$ and the event A = {$x \in$ S: x is a multiple of 3}. Then P(A) is equal to

(1) $\frac{7}{22}$ **(2)** $\frac{1}{5}$ **(3)** $\frac{15}{44}$ **(4)** $\frac{1}{3}$

Q. 14. Let $x = 2$ be a local minima of the function $f(x) = 2x^4 - 18x^2 + 8x + 12$, $x \in (-4, 4)$. If M is local maximum value of the function f in $(-4,4)$, then M =

(1) $18\sqrt{6}-\frac{31}{2}$ **(2)** $18\sqrt{6}-\frac{33}{2}$
(3) $12\sqrt{6}-\frac{33}{2}$ **(4)** $12\sqrt{6}-\frac{31}{2}$

Q. 15. Let f: (0, 1) → ℝ be a function defined by $f(x)=\frac{1}{1-e^{-x}}$, and $g(x) = (f(-x) - f(x))$. Consider two statements:
(I) g is an increasing function in (0,1)
(II) g is one-one in (0,1)
(1) Both (I) and (II) are true
(2) Neither (I) nor (II) is true
(3) Only (I) is true
(4) Only (II) is true

Q. 16. Let $y(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^{16})$. They $y' - y''$ at $x = -1$ is equal to :
(1) 976 **(2)** 944 **(3)** 464 **(4)** 496

Q. 17. The distance of the point P(4, 6, −2) from the line passing through the point (−3, 2, 3) and parallel to a line with direction ratios 3, 3, −1 is equal to :

(1) $\sqrt{14}$ **(2)** 3 **(3)** $\sqrt{6}$ **(4)** $2\sqrt{3}$

Q. 18. Let $x, y, z > 1$ and A = $\begin{bmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 2 & \log_y z \\ \log_z x & \log_z y & 3 \end{bmatrix}$.

Then $|\text{adj}(\text{adj } A^2)|$ is equal to
(1) 2^8 **(2)** 4^8 **(3)** 6^4 **(4)** 2^4

Q. 19. If a_r is the coefficient of x^{10-r} in the Binomial expansion of $(1 + x)^{10}$, then $\sum_{r=1}^{10} r^3\left(\frac{a_r}{a_{r-1}}\right)^2$ is equal to
(1) 5445 **(2)** 3025 **(3)** 4895 **(4)** 1210

Q. 20. Let $y = y(x)$ be the solution curve of the differential equation $\frac{dy}{dx}=\frac{y}{x}(1+xy^2(1+\log_e x))$, $x > 0$, $y(1) = 3$. Then $\frac{y^2(x)}{9}$ is equal to :

(1) $\frac{x^2}{2x^3(2+\log_e x^3)-3}$

(2) $\frac{x^2}{3x^3(1+\log_e x^2)-2}$

(3) $\frac{x^2}{7-3x^3(2+\log_e x^2)}$

(4) $\frac{x^2}{5-2x^3(2+\log_e x^3)}$

Section B

Q. 21. The constant term in the expansion of $\left(2x+\frac{1}{x^7}+3x^2\right)^5$ is

Q. 22. For some $a, b, c \in$ N, let $f(x) = ax - 3$ and $g(x) = x^b + c$, $x \in$ ℝ. If $(f\circ g)^{-1}(x)=\left(\frac{x-7}{2}\right)^{\frac{1}{3}}$ then $(f\circ g)(ac)+(g\circ f)\ b$ is equal to

Q. 23. Let S = {1, 2, 3, 5, 7, 10, 11}. The number of non-empty subsets of S that have the sum of all elements a multiple of 3 , is

Q. 24. Let the equation of the plane passing through the line $x - 2y - z - 5 = 0 = x + y + z - 5$ and parallel to the line $x + y + 2z - 7 = 2x + 3y + z - 2$ be $ax + by + cz = 65$. Then the distance of the point (a, b, c) from the plane $2x + 2y - z + 16 = 0$ is

Q. 25. If the sum of all the solutions of $\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3}$, $-1 < x < 1$, $x \neq 0$, is $\alpha-\frac{4}{\sqrt{3}}$ then α is equal to

Q. 26. The vertices of a hyperbola H are (±6, 0) and its eccentricity is $\frac{\sqrt{5}}{2}$. Let N be the normal to H at a point in the first quadrant and parallel to the line $\sqrt{2}x+y=2\sqrt{2}$. If d is the length of the line segment of N between H and the y-axis then d^2 is equal to

Q. 27. Let x and y be distinct integers where $1 \le x \le 25$ and $1 \le y \le 25$. Then, the number of ways of choosing x and y, such that $x + y$ is divisible by 5, is

Q. 28. Let $S = \{a : \log_2(9^{2\alpha-4} + 13) - \log_2\left(\frac{5}{2}.3^{2\alpha-4} + 1\right) = 2\}$.

Then the maximum value of β for which the equation $x^2 - 2\left(\sum_{\alpha\in S}\alpha\right)^2 x + \sum_{\alpha\in S}(\alpha+1)^2\beta = 0$ has real roots, is

Q. 29. It the area enclosed by the parabolas $P_1 : 2y = 5x^2$ and $P_2 : x^2 - y + 6 = 0$ is equal to the area enclosed by P_1 and $y = \alpha x$, $\alpha > 0$, then a^3 is equal to

Q. 30. Let A_1, A_2, A_3 be the three A.P. with the same common difference d and having their first terms as A, A + 1, A + 2, respectively. Let a b, c be the 7^{th}, 9^{th}, 17^{th} terms of A_1, A_2, A_3, respectively such that $\begin{vmatrix} a & 7 & 1 \\ 2b & 17 & 1 \\ c & 17 & 1 \end{vmatrix} + 70 = 0$

If $a = 29$, then the sum of first 20 terms of an AP whose first term is $c - a - b$ and common difference is $\frac{d}{12}$, is equal to.

Answer Key

Q. No.	Answer	Topic Name	Chapter Name
1	(4)	Equation of circle	Circle
2	(4)	Tangent to a Parabola	Parabola
3	(2)	Triple Product	Vector Algebra
4	(3)	Scalar and Vector Product	Vector Algebra
5	(4)	Algebra of Complex Numbers	Complex Numbers
6	(1)	Measures of Dispersion	Statistics
7	(4)	Solution of linear equation	Matrices and Determinants
8	(1)	Limit	Limit, Continuity and Differentiability
9	(1)	Tautology and Contradiction	Mathematical Reasoning
10	(4)	Line	Three Dimensional Geometry
11	(4)	Integration by substitution	Indefinite Integration
12	(2)	Maxima and Minima	Application of Derivatives
13	(4)	Basics of Probability	Probability
14	(3)	Maxima and Minima	Application of Derivatives
15	(1)	Monotonicity	Application of Derivatives
16	(4)	Higher order derivatives	Differentiation
17	(1)	Line and a Point	Three Dimensional Geometry
18	(1)	Adjoint of a matrix	Matrices and Determinants
19	(4)	Properties of Binomial coefficients	Binomial Theorem
20	(4)	Linear Differential Equation	Differential Equations
21	[1080]	Binomial Theorem for Positive Integral Index	Binomial Theorem
22	[2039]	Composition of functions	Function
23	[43]	Basics of set	Set Theory
24	[9]	Planes in 3D	Three Dimensional Geometry
25	[2]	Properties of Inverse trigonometric functions	Inverse Trigonometric Functions
26	[216]	Tangent and Normal	Hyperbola
27	[120]	Combination	Permutation and Combination
28	[25]	Nature of roots	Quadratic Equations
29	[600]	Area Bounded by Curves	Area under Curves
30	[495]	Arithmetic Progressions	Sequences and Series

Solutions

Section A

1. **Option (4) is correct.**

Given: Equation of circle is $x^2 + y^2 - 2x = 0$

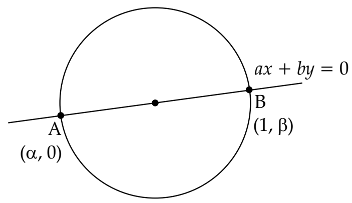
Equation of line is $ax + by = 0$

$\because$ Point A lies on given line

$\therefore a\alpha + b(0) = 0$

$\Rightarrow \alpha = 0$

And point B lies on line and circle



So, $a + b\beta = 0$

And $1 + \beta^2 - 2 = 0$

$\Rightarrow \beta = 1$

So, A = (0, 0) and B = (1, 1)

Now, centre of circle as AB diameter is $\left(\frac{1}{2}, \frac{1}{2}\right)$ and radius = $\frac{1}{\sqrt{2}}$

Now, for image of $\left(\frac{1}{2}, \frac{1}{2}\right)$ in $x + y + 2 = 0$, we get

$$\frac{x-\frac{1}{2}}{1} = \frac{y-\frac{1}{2}}{1} = \frac{-2\left(\frac{1}{2}+\frac{1}{2}+2\right)}{1^2+1^2}$$

$\Rightarrow x = \frac{-5}{2}, y = \frac{-5}{2}$

$\therefore$ Equation of required circle is

$$\left(x+\frac{5}{2}\right)^2 + \left(y+\frac{5}{2}\right)^2 = \frac{1}{2}$$

$\Rightarrow x^2 + y^2 + 5x + 5y + 12 = 0$

HINT:

(1) Find α and β by satisfying points A and B in given equation of line and circle.

(2) Mirror image of point A(x_1, y_1) in line $ax + by + c = 0$ is given by

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{-2(ax_1+by_1+c)}{a^2+b^2}$$

2. **Option (4) is correct.**

Equation of tangent to $y^2 = \frac{x}{2}$ is given by in slope form, $y = mx + \frac{1}{8m}$ ...(i)

Equation of tangent to $y^2 = x - 1$ in slope form is given by $y = (x-1)m + \frac{1}{4m}$ ...(ii)

$\because$ Equation (i) and equation (ii) represents the same equation.

$\therefore \frac{1}{8m} = -m + \frac{1}{4m}$

$\therefore 1 = -8m^2 + 2$

$\Rightarrow m^2 = \frac{1}{8} \Rightarrow m = \pm\frac{1}{2\sqrt{2}}$

$\because m > 0$ So, $m = \frac{1}{2\sqrt{2}}$

So, equation of common tangent to both given curve is $y = \frac{x}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$

$\Rightarrow 2\sqrt{2}y - x - 1 = 0$

Now, distance of the point $(6, -2\sqrt{2})$ from $2\sqrt{2}y - x - 1 = 0$ is

$$d = \left|\frac{2\sqrt{2}(-2\sqrt{2}) - 6 - 1}{\sqrt{(2\sqrt{2})^2 + (-1)^2}}\right|$$

$\Rightarrow d = \left|\frac{-15}{3}\right| = 5$

3. **Option (2) is correct.**

Given: $\vec{b}.\vec{c} = 0, \vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{b}-\vec{c}}{2}$ and $\vec{b}.\vec{d} = \vec{a}.\vec{b}$

$\because \vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{b}-\vec{c}}{2}$

$\Rightarrow (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = \frac{\vec{b}}{2} - \frac{\vec{c}}{2}$

$\Rightarrow [(\vec{a}.\vec{c}) - \frac{1}{2}]\vec{b} - \left[(\vec{a}.\vec{b}) - \frac{1}{2}\right]\vec{c} = 0$

$\Rightarrow \vec{a}.\vec{c} - \frac{1}{2} = 0$ and $\vec{a}.\vec{b} - \frac{1}{2} = 0$

$\Rightarrow \vec{a}.\vec{c} = \frac{1}{2}$ and $\vec{a}.\vec{b} = \frac{1}{2}$

Now, $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = \vec{a}.(\vec{b} \times (\vec{c} \times \vec{d}))$

$= \vec{a}.[(\vec{b}.\vec{d})\vec{c} - (\vec{b}.\vec{c})\vec{d}]$

$= \vec{a}.\left[\frac{1}{2}\vec{c} - 0\right] = \frac{1}{2}\vec{a}.\vec{c} = \frac{1}{4}$

4. **Option (3) is correct.**

Given : $\vec{c} = 5\hat{i} + 4\hat{j} + 3\hat{k}$

And vector $\vec{a} = -\hat{i} + 2\hat{j} + \hat{k}$ is rotated through a right angle, and passing through the y-axis in its way.

Let $\vec{b} = m\vec{a} + n\hat{j}$

$\because \vec{b} \perp \vec{a}$

$\therefore \vec{b}.\vec{a} = 0$

$\Rightarrow (m\vec{a} + n\hat{j}).\vec{a} = 0$

$\Rightarrow m|\vec{a}|^2 + n(\hat{j}).(-\hat{i} + 2\hat{j} + \hat{k}) = 0$

$\Rightarrow m(6) + 2n = 0 \Rightarrow 6m + 2n = 0$ ...(i)

So, $\vec{b} = m\vec{a} + (-3m)\hat{j}$

$\Rightarrow \vec{b} = m(-\hat{i} + 2\hat{j} + \hat{k}) - 3m\hat{j}$

$\Rightarrow \vec{b} = m(-\hat{i} - \hat{j} + \hat{k})$

Also, $|\vec{b}| = |\vec{a}|$

$\Rightarrow |\vec{b}|^2 = |\vec{a}|^2 \Rightarrow m = \pm\sqrt{2}$

Case-1: When $m = \sqrt{2}$

$\vec{b} = \sqrt{2}(-\hat{i} - \hat{j} + \hat{k})$

$3\vec{a} + \sqrt{2}\vec{b} = 3(-\hat{i} + 2\hat{j} + \hat{k}) + \sqrt{2}\left(\sqrt{2}(-\hat{i} - \hat{j} + \hat{k})\right)$

$= -5\hat{i} + 4\hat{j} + 5\hat{k}$

So, projection of $3\vec{a} + \sqrt{2}\vec{b}$ on $\vec{c} = \dfrac{(3\vec{a} + \sqrt{2}\vec{b}).\vec{c}}{|\vec{c}|}$

$= \dfrac{(-5\hat{i} + 4\hat{j} + 5\hat{k}).(5\hat{i} + 4\hat{j} + 3\hat{k})}{\sqrt{25 + 16 + 9}}$

$= \dfrac{-25 + 16 + 15}{\sqrt{50}} = \dfrac{3\sqrt{2}}{5}$

Case 2: When $m = -\sqrt{2}$

$\vec{b} = -\sqrt{2}(-\hat{i} - \hat{j} + \hat{k})$

So, $3\vec{a} + \sqrt{2}\vec{b} = -\hat{i} + 8\hat{j} + \hat{k}$

Now, projection of $3\vec{a} + \sqrt{2}\vec{b}$ on $\vec{c}$

$= \dfrac{(-\hat{i} + 8\hat{j} + \hat{k}).(5\hat{i} + 4\hat{j} + 3\hat{k})}{\sqrt{50}} = \dfrac{-5 + 32 + 3}{\sqrt{50}}$

$= \dfrac{30}{\sqrt{50}} = 3\sqrt{2}$

5. **Option (4) is correct.**

Given: $z_1 = 2 + 3\vec{i}$, $z_2 = 3 + 4\vec{i}$

And S = $\{z \in C : |z - z_1|^2 - |z - z_2|^2 = |z_1 - z_2|^2\}$

Let $z = x + iy$

Now, $(z - z_1) = (x - 2) + i(y - 3)$

$(z - z_2) = (x - 3) + i(y - 4)$

$(z_1 - z_2) = -1 - i$

$\because |z - z_1|^2 - |z - z_2|^2 = |z_1 - z_2|^2$

$\Rightarrow [(x-2)^2 + (y-3)^2] - [(x-3)^2 + (y-4)^2] = 1 + 1$

$\Rightarrow x + y = 7$

$\Rightarrow \dfrac{x}{7} + \dfrac{y}{7} = 1$

So, S represents a straight line with the sum of its intercepts on coordinate axis equals 14.

HINT:

Assume $z = x + iy$ and solved further using the concept of modulus of complex number.

6. **Option (1) is correct.**

Given : (mean) = 10 and variance = 4

Let number of observations be n.

So, $\dfrac{\sum x_i}{n} = 10$

Also given that marks of one student is increased from 8 to 12.

And $(\text{mean})_{\text{new}} = 10.2$

$\Rightarrow \dfrac{\Sigma x_i - 8 + 12}{n} = 10.2$

$\Rightarrow \Sigma x_i + 4 = (10.2)\, n$

$\Rightarrow 10n + 4 = 10.2\, n$

$\Rightarrow n = 20$

$\because$ Variance = 4

$\Rightarrow \dfrac{\Sigma x_i^2}{n} - (\bar{x})^2 = 4$

$\Rightarrow \dfrac{\Sigma x_i^2}{20} - 100 = 4 \quad \Rightarrow \Sigma x_i^2 = 2080$

Now, after change $\Sigma x_i^2 = 2080 - 64 + 144 = 2160$

So, $(\text{variance})_{\text{new}} = \dfrac{2160}{20} - (10.2)^2$

$= 108 - 104.04 = 3.96$

HINT:

Use mean = $\dfrac{\Sigma x_i}{n}$ and variance = $\dfrac{\Sigma x_i^2}{n} - (\bar{x})^2$

7. **Option (4) is correct.**

Given : System of linear equations

$ax + 2ay - 3az = 1$

$(2a + 1)x + (2a + 3)y + (a + 1)z = 2$

$(3a + 5)x + (a + 5)y + (a + 2)z = 3$

Now, $\Delta = \begin{vmatrix} a & 2a & -3a \\ 2a+1 & 2a+3 & a+1 \\ 3a+5 & a+5 & a+2 \end{vmatrix}$

$\Rightarrow \Delta = a\begin{vmatrix} 1 & 2 & -3 \\ 2a+1 & 2a+3 & a+1 \\ 3a+5 & a+5 & a+2 \end{vmatrix}$

Applying $C_2 \to C_2 - 2C_1$ and $C_3 \to C_3 + 3C_1$, we get

$\Delta = a\begin{vmatrix} 1 & 0 & 0 \\ 2a+1 & -2a+1 & 7a+4 \\ 3a+5 & -5a-5 & 10a+17 \end{vmatrix}$

$\Rightarrow \Delta = a\,[(-2a + 1)(10a + 17) + (5a + 5)(7a + 4)]$

$\Rightarrow \Delta = a\,(15a^2 + 31a + 37)$

For unique solution, $\Delta \neq 0$

$\Rightarrow a \neq 0$ So, $S_1 = R - \{0\}$

For infinite many solutions, $\Delta = 0$

$\Rightarrow a = 0$

But $a \in R - \{0\}$ So, $S_2 = \phi$

HINT:

Recall cramer rule and use for unique solution, $\Delta \neq 0$

For infinite solutions, $\Delta = 0$

8. **Option (1) is correct.**

Let A =

$$\lim_{n\to\infty} \frac{1 + 2 - 3 + 4 + 5 - 6 + \ldots + (3n-2) + (3n-1) - 3n}{\sqrt{2n^4 + 4n + 3} - \sqrt{n^4 + 5n + 4}}$$

$$\Rightarrow \text{A} = \lim_{n\to\infty} \frac{(1+2+3+....+3n)-2(3+6+9+....+3n)}{\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4}}$$

$$\Rightarrow \text{A} = \lim_{n\to\infty} \frac{\frac{3n(3n+1)}{2}-\frac{6n(n+1)}{2}}{\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4}}$$

$$\Rightarrow \text{A} = \lim_{n\to\infty} \frac{\frac{3}{2}n(n-1)\left[\sqrt{2n^4+4n+3}+\sqrt{n^4+5n+4}\right]}{n^4-n-1}$$

$$\Rightarrow \text{A} = \lim_{n\to\infty} \frac{\frac{3}{2}n^4\left(1-\frac{1}{n}\right)\left[\sqrt{2+\frac{4}{n^3}+\frac{3}{n^4}}+\sqrt{1+\frac{5}{n^3}+\frac{4}{n^4}}\right]}{n^4\left[1-\frac{1}{n^3}-\frac{1}{n^4}\right]}$$

$$\Rightarrow \text{A} = \frac{\frac{3}{2}(1-0)[\sqrt{2+0}+\sqrt{1}]}{[1-0-0]}$$

$$\Rightarrow \text{A} = \frac{3}{2}(1+\sqrt{2})$$

9. Option (1) is correct.

Given statement is $(p \wedge (\sim q)) \Rightarrow (p \Rightarrow (\sim q))$

As we know $\text{A} \Rightarrow \text{B} = \sim \text{A} \vee \text{B}$

So, $p \Rightarrow (\sim q) = \sim p \vee (\sim q)$

Now, $(p \wedge (\sim q)) \Rightarrow (p \Rightarrow (\sim q))$

$= \sim [p \wedge (\sim q)] \vee [\sim p \vee (\sim q)]$

$= [\sim p \vee q] \vee [\sim p \vee (\sim q)]$

$= \sim p \vee q \vee (\sim q) = \sim p \vee \text{T} = \text{T}$

So, given statement is tautology.

HINT:

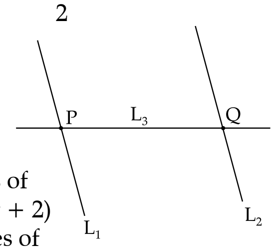
(1) Use $\text{A} \Rightarrow \text{B} = \sim \text{A} \vee \text{B}$

(2) $\text{A} \vee \text{T} = \text{T}$

10. Option (4) is correct.

Given lines $\text{L}_1 : \frac{x-1}{2} = \frac{y-3}{1} = \frac{z-2}{2}$

$\text{L}_2 : \frac{x-2}{1} = \frac{y-2}{2} = \frac{z-3}{3}$



And direction ratios of $\text{L}_3 = 1, -1, -2$

Let parametric coordinates of point P = $(2k + 1, k + 3, 2k + 2)$

And parametric coordinates of point Q = $(\lambda + 2, 2\lambda + 2, 3\lambda + 3)$

Now, direction ratios of PQ = $2k - \lambda - 1, k - 2\lambda + 1, 2k - 3\lambda - 1$

$\because$ direction ratios of $\text{L}_3 = 1, -1, -2$

So, $\frac{2k-\lambda-1}{1} = \frac{k-2\lambda+1}{-1} = \frac{2k-3\lambda-1}{-2}$

$\Rightarrow 2k - \lambda - 1 = -k + 2\lambda - 1$

and $(2k - \lambda - 1)(-2) = 2k - 3\lambda - 1$

$\Rightarrow 3k - 3\lambda = 0$ and $6k - 5\lambda = 3$

$\Rightarrow k = \lambda$ and $6k - 5\lambda = 3$

$\Rightarrow k = \lambda = 3$

So, P = (7, 6, 8) and Q = (5, 8, 12)

Now, $\text{PQ} = \sqrt{(7-5)^2+(6-8)^2+(8-12)^2}$

$= \sqrt{4+4+16} \Rightarrow \text{PQ} = 2\sqrt{6}$

HINT:

Assume parametric coordinates of point P and Q and then solve further using the concept of direction ratios.

11. Option (4) is correct.

Given, $f(x) = \int \frac{2x}{(x^2+1)(x^2+3)}dx$

Let $x^2 = u \Rightarrow 2x\, dx = du$

$$\Rightarrow f(x) = \int \frac{du}{(u+1)(u+3)}$$

$$\Rightarrow f(x) = \int \left(\frac{1}{u+1} - \frac{1}{u+3}\right)\frac{du}{2}$$

$$\Rightarrow f(x) = \frac{1}{2}\ln\left(\frac{u+1}{u+3}\right) + c$$

$$\Rightarrow f(x) = \frac{1}{2}\ln\left(\frac{x^2+1}{x^2+3}\right) + c$$

Put $x = 3$ in above equation, we get

$$f(3) = \frac{1}{2}\ln\left(\frac{10}{12}\right) + c$$

$$\Rightarrow \frac{1}{2}\ln\left(\frac{5}{6}\right) = \frac{1}{2}\ln\left(\frac{5}{6}\right) + c \Rightarrow c = 0$$

$$\text{So, } f(x) = \frac{1}{2}\ln\left(\frac{x^2+1}{x^2+3}\right)$$

$$\Rightarrow f(4) = \frac{1}{2}\ln\left(\frac{17}{19}\right) \Rightarrow f(4) = \frac{1}{2}[\ln 17 - \ln 19]$$

HINT:

Substitute $x^2 = u$ and solved further.

12. Option (2) is correct.

Given, $f(x) = \int_0^2 e^{|x-t|}\,dt$

Case-1: When $x < 0$

$$f(x) = \int_0^2 e^{-(x-t)}\,dt$$

$$\Rightarrow f(x) = \int_0^2 e^{-x}.e^{t}\,dt \Rightarrow f(x) = e^{-x}\left[e^t\right]_0^2$$

$\Rightarrow f(x) = e^{-x}(e^2 - 1)$

Case-2: When $x \in [0, 2]$

$$f(x) = \int_0^2 e^{|x-t|}\,dt$$

$$\Rightarrow f(x) = \int_0^x e^{x-t}\,dt + \int_x^2 e^{-(x-t)}dt$$

$$\Rightarrow f(x) = e^x\left[-e^{-t}\right]_0^x + e^{-x}\left[e^t\right]_x^2$$

$$\Rightarrow f(x) = e^x\left[-e^{-x}+1\right] + e^{-x}[e^2 - e^x]$$

$\Rightarrow f(x) = e^x + e^{2-x} - 2$

Case-3: When $x > 2$

$$f(x) = \int_0^2 e^{x-t}\,dt$$

$$\Rightarrow f(x) = e^x\left[-e^{-t}\right]_0^2 \Rightarrow f(x) = e^x\left[1 - e^{-2}\right]$$

So, $[f(x)]_{\min} = e^2 - 1;\ x < 0$
$= 2(e - 1)\ ;\ x \in [0, 2] = e^2 - 1;\ x > 2$
So, minimum value of $f(x) = 2\,(e - 1)$

13. Option (4) is correct.

Let two positive integers be a and b.

Given, $S = \{x \in z : x(66 - x) \geq \frac{5}{9}M\}$:

Where max $M = (ab)$
$A = \{x \in S : x$ is a multiple of 3$\}$
As we know for positive numbers, A.M. ≥ G.M.

So, $\frac{a+b}{2} \geq \sqrt{ab}$

$\Rightarrow \sqrt{ab} \leq \frac{66}{2}$ $\qquad \{\because a + b = 66\}$

$\Rightarrow (a\,b)_{\max} = 33^2 \Rightarrow M = (33)^2$

$\because x(66 - x) \geq \frac{5}{5}M \Rightarrow x(66 - x) \geq \frac{5}{9}(33)^2$

$\Rightarrow x\,(66 - x) \geq 605 \Rightarrow (x - 11)\,(x - 55) \leq 0$

$\Rightarrow x \in [11, 55]$

$\Rightarrow S = [11, 12, 13, \ldots, 55]$

$\Rightarrow A = [12, 15, 18, \ldots, 54]$

So, $n(S) = 45$ and $n(A) = 15$

Now, $P(A) = \frac{n(A)}{n(S)} = \frac{15}{45} = \frac{1}{3}$

14. Option (3) is correct.

Given, $f(x) = 2x^4 - 18x^2 + 8x + 12;\ x \in (-4, 4)$

$\Rightarrow f'(x) = 8x^3 - 36x + 8$

$\Rightarrow f'(x) = 4\,(2x^3 - 9x + 2)$

$\because x = 2$ is local minima of $f(x)$
$\therefore (x - 2)$ is a factor of $f'(x) = 4\,(2x^3 - 9x + 2)$
So, $f'(x) = 4\,(x - 2)\,(2x^2 + 4x - 1)$

$$\Rightarrow f'(x) = 4(x-2)\left[x - \left(\frac{-2+\sqrt{6}}{2}\right)\right]\left[x - \left(\frac{-2-\sqrt{6}}{2}\right)\right]$$

$-\quad \frac{-2-\sqrt{6}}{2} \quad + \quad \frac{-2+\sqrt{6}}{2} \quad - \quad 2 \quad +$

$\because$ Sign of $f'(x)$ changes from +ve to –ve of $x = \frac{-2+\sqrt{6}}{2}$.

So, $f(x)$ has local maxima at $x = \frac{-2+\sqrt{6}}{2}$

Now, $$f\left(\frac{-2+\sqrt{6}}{2}\right) = 2\left(\frac{-2+\sqrt{6}}{2}\right)^4 - 18\left(\frac{-2+\sqrt{6}}{2}\right)^2 + 8\left(\frac{-2+\sqrt{6}}{2}\right) + 12$$

$$= 2\left[\frac{5-2\sqrt{6}}{2}\right]^2 - 18\left[\frac{5-2\sqrt{6}}{2}\right] + 8\left(-1 + \sqrt{\frac{3}{2}}\right) + 12$$

$$= 2\left[\frac{49-20\sqrt{6}}{4}\right] - 45 + 18\sqrt{6} - 8 + 4\sqrt{6} + 12$$

$$= \frac{49}{2} - 10\sqrt{6} - 41 + 22\sqrt{6} \quad = 12\sqrt{6} - \frac{33}{2}$$

So, $M = 12\sqrt{6} - \frac{33}{2}$

15. Option (1) is correct.

Given, $f(x) = \frac{1}{1 - e^{-x}}$

And $g(x) = f(-x) - f(x)$

So, $g(x) = \frac{1}{1-e^x} - \frac{1}{1-e^{-x}}$

$$\Rightarrow g(x) = \frac{1}{1-e^x} - \frac{e^x}{e^x - 1} \Rightarrow g(x) = \frac{1+e^x}{1-e^x}$$

$$\Rightarrow g'(x) = \frac{(1-e^x)\frac{d}{dx}(1+e^x) - (1+e^x)\frac{d}{dx}(1-e^x)}{(1-e^x)^2}$$

$$\Rightarrow g'(x) = \frac{(1-e^x)e^x - (1+e^x)(-e^x)}{(1-e^x)^2}$$

$$\Rightarrow g'(x) = \frac{2e^x}{(1-e^x)^2} > 0$$

$\Rightarrow g(x)$ is increasing function and one-one function.

16. Option (4) is correct.

Given, $y(x) = (1 + x)\,(1 + x^2)\,(1 + x^4)\,(1 + x^6)\,(1 + x^{16})$

$$\Rightarrow y(x) = \frac{(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})}{(1-x)}$$

$$\Rightarrow y(x) = \frac{1-x^{32}}{1-x} \Rightarrow y(-1) = 0$$

$$\because y = \frac{1-x^{32}}{1-x} \Rightarrow y(1-x) = 1 - x^{32}$$

Differentiate above equation w.r.t. x, we get
$y'\,(1 - x) + y(-1) = -32x^{31}$
$\Rightarrow y'(1 - x) - y = -32\,x^{31}$ ...(i)
Put $x = -1$ in above equation, we get
$y'(2) - 0 = -32\,(-1)$
$\Rightarrow [y'] = 16$
Differentiate equation (i) w.r.t. x, we get
$y''(1 - x) - y' - y' = (-32)\,(31)\,x^{30}$
$\Rightarrow y''\,(1 - x) - 2y' = -992\,x^{30}$
Put $x = -1$ in above equation, we get
$y''(2) - 2\,(16) = -992$
$[y'']_{x=-1} = -480$
$\Rightarrow [y' - y'']_{x=-1} = 16 + 480 = 496$

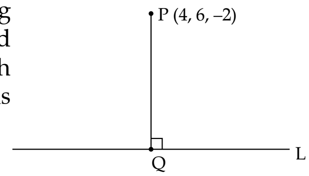
HINT:

Multiply and divide the expression of y by $(1 - x)$ and solved further.

17. Option (1) is correct.

Given, Point P = (4, 6, –2)

Equation of line passing through (–3, 2, 3) and parallel to a line with direction ratios 3, 3, –1 is given by



$$L: \frac{x+3}{3} = \frac{y-2}{3} = \frac{z-3}{-1} = m$$

Let coordinates of point Q in parametric form be $(3m - 3, 3m + 2, -m + 3)$

Now, direction ratios of PQ = $3m - 7, 3m - 4, -m + 5$

$\because$ PQ $\perp$ Line L

$\therefore (3m - 7)(3) + (3m - 4)(3) + (5 - m)(-1) = 0$

$\Rightarrow 19m = 38 \Rightarrow m = 2$

$\therefore$ Q = (3, 8, 1)

Now, PQ = $\sqrt{(4-3)^2+(6-8)^2+(-2-1)^2} = \sqrt{14}$

18. Option (1) is correct.

Given, $A = \begin{bmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 2 & \log_y z \\ \log_z x & \log_z y & 3 \end{bmatrix}$

Now, adj (adj A^2) = $|A^2|^{(3-2)} A^2$

$\Rightarrow$ adj (adj A^2) = $|A^2|\, A^2$

$\Rightarrow$ |adj (adj A^2)| = $|A^2|\, |A|^2 = |A^2|^4$

$\Rightarrow$ |adj (adj A^2)| = $|A|^8$

Now, $|A| = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 2 & \log_y z \\ \log_z x & \log_z y & 3 \end{vmatrix}$

$$\Rightarrow |A| = \begin{vmatrix} 1 & \frac{\ln y}{\ln x} & \frac{\ln z}{\ln x} \\ \frac{\ln x}{\ln y} & 2 & \frac{\ln z}{\ln y} \\ \frac{\ln x}{\ln z} & \frac{\ln y}{\ln z} & 3 \end{vmatrix}$$

$$\Rightarrow |A| = \frac{1}{\ln x \ln y \ln z}\begin{vmatrix} \ln x & \ln y & \ln z \\ \ln x & 2\ln y & \ln z \\ \ln x & \ln y & 3\ln z \end{vmatrix}$$

$$\Rightarrow |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{vmatrix}$$

$\Rightarrow |A| = 1(6-1) - 1(3-1) + 1(1-2)$

$\Rightarrow |A| = 2$

So, |adj (adj A^2)| = $|A|^8 = 2^8$

HINT:

(1) Use adj (adj A) = $|A|^{n-2} A$

(2) Use |adj A| = $|A|^{n-1}$

19. Option (4) is correct.

Given a_r = coefficient of x^{10-r} in $(1 + x)^{10}$

Now, general term of binomial expansion of $(1 + x)^{10}$ is $T_{r+1} = {}^{10}C_r x^r$

$\therefore a_r = {}^{10}C_{10-r}$

$\Rightarrow a_r = {}^{10}C_r$

Now, $\frac{a_r}{a_{r-1}} = \frac{{}^{10}C_r}{{}^{10}C_{r-1}} = \frac{10-r+1}{r} = \frac{11-r}{r}$

So, $\sum_{r=1}^{10} r^3\left(\frac{a_r}{a_{r-1}}\right)^2 = \sum_{r=1}^{10} r^3\left(\frac{11-r}{r}\right)^2$

$$= \sum_{r=1}^{10} r(11-r)^2 = \sum_{r=1}^{10} r(r^2+121-22r)$$

$$= \sum_{r=1}^{10}\left(r^3 - 22r^2 + 121r\right)$$

$$= \sum_{r=1}^{10} r^3 - 22\sum_{r=1}^{10} r^2 + 121\sum_{r=1}^{10} r$$

$$= \left[\frac{10(11)}{2}\right]^2 - 22\left[\frac{10(11)(21)}{6}\right] + 121\left(\frac{10\times 11}{2}\right)$$

$= 1210$

20. Option (4) is correct.

Given, $\frac{dy}{dx} = \frac{y}{x}(1 + xy^2(1+\ln x)), y(1) = 3$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + y^3(1+\ln x)$$

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} - \frac{1}{y^2}\left(\frac{1}{x}\right) = 1 + \ln x$$

Let $-\frac{1}{y^2} = u \Rightarrow \frac{1}{y^3}\frac{dy}{dx} = \frac{1}{2}\frac{du}{dx}$

$$\Rightarrow \frac{1}{2}\frac{du}{dx} + \frac{u}{x} = 1 + \ln x$$

$\Rightarrow \frac{du}{dx} + \frac{2u}{x} = 2(1+\ln x)$, which is linear differential equation,

Now, I.F. = $e^{\int \frac{2}{x}dx}$

$\Rightarrow$ I.F. = $e^{2\ln x} = x^2$

So, solution of given differential equation is

$u\,(\text{I.F.}) = \int 2(1+\ln x)(\text{I.F.})\,dx + C$

$$\Rightarrow ux^2 = \int 2x^2(1+\ln x)\,dx + C$$

$$\Rightarrow ux^2 = 2(1+\ln x)\frac{x^3}{3} - 2\int\left(\frac{1}{x}\right)\cdot\frac{x^3}{3}dx + C$$

$$\Rightarrow ux^2 = \frac{2x^3}{3}(1+\ln x) - \frac{2x^3}{9} + C$$

$$\Rightarrow -\frac{1}{y^2}x^2 = \frac{2x^3}{3}(1+\ln x) - \frac{2x^3}{9} + C$$

Put $x = 1$ in above equation, we get

$$-\frac{1}{9}(1) = \frac{2}{3}(1+0) - \frac{2}{9} + C$$

$$\Rightarrow C = -\frac{5}{9}$$

$$\therefore -\frac{x^2}{y^2} = \frac{2x^3}{3}(1+\ln x) - \frac{2x^3}{9} - \frac{5}{9}$$

$$\Rightarrow \frac{x^2}{y^2} = -\frac{1}{9}[6x^3(1+\ln x) - 2x^3 - 5]$$

$$\Rightarrow \frac{x^2}{y^2} = \frac{1}{9}[5 - 4x^3 - 6x^3 \ln x]$$

$$\Rightarrow \frac{x^2}{y^2} = \frac{1}{9}[5 - 2x^3(2+\ln x^3)]$$

$$\Rightarrow \frac{y^2}{9} = \frac{x^2}{5 - 2x^3(2+\ln x^3)}$$

Section B

21. Correct answer is [1080].

Let A = constant term in $\left(2x + \frac{1}{x^7} + 3x^2\right)^5$

$\Rightarrow$ A = constant term in $\frac{1}{x^{35}}\left(2x^8 + 1 + 3x^9\right)^5$

$\Rightarrow$ A = coefficient of x^{35} in $(1 + x^8(3x+2))^5$

$\Rightarrow$ A = coefficient of x^{35} in ${}^5C_4\,[x^8(3x+2)]^4$

$\Rightarrow$ A = coefficienf of x^3 in ${}^5C_4\,(3x+2)^4$

$\Rightarrow$ A = ${}^5C_4 \times {}^4C_3\,(3)^3\,(2)^1$

$\Rightarrow$ A = 5 × 4 × 27 × 2 $\Rightarrow$ A = 1080

22. Correct answer is [2039].

Given, $f(x) = ax - 3$ and $g(x) = x^b + c$, $x \in R$

And $(fog)^{-1}(x) = \left(\frac{x-7}{2}\right)^{\frac{1}{3}}$

Now, $fog = f[g(x)]$

$= a[x^b + c] - 3$

$= ax^b + ac - 3$

$$\Rightarrow (fog)^{-1}(x) = \left(\frac{x+3-ac}{a}\right)^{\frac{1}{b}}$$

$$\Rightarrow \left(\frac{x-7}{2}\right)^{\frac{1}{3}} = \left(\frac{x+3-ac}{a}\right)^{\frac{1}{b}}$$

$\Rightarrow a = 2, b = 3, c = 5$

So, $f(x) = 2x - 3$ and $g(x) = x^3 + 5$

Now, $fog(ac) + gof(b) = fog(10) + gof(3)$

$= f[g(10)] + g[f(3)]$

$= f[1005] + g[3] = 2(1005) - 3 + 3^3 + 5$

$= 2007 + 32 = 2039$

HINT:

Use $fog(x) = f[g(x)]$ and recall method for finding inverse of a function.

23. Correct answer is [43].

S = {1, 2, 3, 5, 7, 10, 11}

For sum of elements to be multiple of 3, elements can be of type $3k$, $3k + 1$, $3k + 2$

$3k \in \{3\}$, $3k + 1 \in \{1, 7, 10\}$, $3k + 2 \in \{2, 5, 11\}$

Subsets having one element = $\{3k\}$

No. of subsets = 1

Subets having two elements = $\{3k + 1, 3k + 2\}$

No. of subsets = ${}^3C_1 \times {}^3C_1 = 9$

Subsets having three elements = $\{3k, 3k + 1, 3k + 2\}$ or $\{3k + 1, 3k + 1, 3k + 1\}$ or $\{3k + 2, 3k + 2, 3k + 2\}$

No. of subsets = $(1 \times {}^3C_1 \times {}^3C_1) + (1) + (1)$

$= 9 + 1 + 1 = 11$

Subsets having four elements = $\{3k, 3k + 1, 3k + 1, 3k + 1\}$ or $\{3k, 3k + 2, 3k + 2, 3k + 2\}$ or $\{3k + 1, 3k + 2, 3k + 1, 3k + 2\}$

No. of subsets = $(1 \times {}^3C_3) + (1 \times {}^3C_3) + ({}^3C_2 \times {}^3C_2)$

$= 1 + 1 + (3 \times 3) = 11$

Subsets having five elements = $\{3k, 3k + 1, 3k + 2, 3k + 1, 3k + 2\}$

No. of subsets = $1 \times {}^3C_2 \times {}^3C_2 = (1 \times 3 \times 3) = 9$

Subsets having six elements= $\{3k + 1, 3k + 2, 3k + 1, 3k + 2, 3k + 1, 3k + 2\}$

No. of subsets = ${}^3C_3 \times {}^3C_3 = 1$

Subsets having seven elements = S........

No. of subset = 1

Total no. of subsets = 1 + 9 + 11 + 11 + 9 + 1 + 1 = 43

HINT:

For sum of elements to be multiple of 3, elements can be of type $3k$, $3k + 1$, $3k + 2$

$3k \in \{3\}$, $3k + 1 \in \{1, 7, 10\}$, $3k + 2 \in \{2, 5, 11\}$

24. Correct answer is [9].

Let the equation of plane is

$(x - 2y - z - 5) + k(x + y + 3z - 5) = 0$ ...(i)

$\because$ Plane is parallel to the line $x + y + 2z - 7 = 0 = 2x + 3y + z - 2$

So, vector along the line = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$

$= \hat{i}(1-6) - \hat{j}(1-4) + \hat{k}(3-2)$

$= -5\hat{i} + 3\hat{j} + \hat{k}$

So, direction ratios of line = –5, 3, 1

$\because$ Plane is parallel to the line

$\therefore -5(1 + k) + 3(k - 2) + 1(3k - 1) = 0$

$\Rightarrow k = 12$

So, required plane is $13x + 10y + 35z = 65$

$\therefore a = 13, b = 10, c = 35$

Now, distance of (13, 10, 35) from $2x + 2y - z + 16 = 0$ is

$$d = \left|\frac{2(13) + 2(10) - 35 + 16}{\sqrt{4+4+1}}\right|$$

$\Rightarrow d = 9$

25. Correct answer is [2].

Given,

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3}, -1 < x < 1, x \neq 0$$

Case-1: $x \in (-1, 0)$

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{1-x^2}\right) = -\frac{\pi}{3}$$

$$\Rightarrow x = -\frac{1}{\sqrt{3}}$$

Case-2: $x \in (0, 1)$

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{6}$$

$$\Rightarrow x = 2 - \sqrt{3}$$

So, sum of all solutions of given equation is

$$-\frac{1}{\sqrt{3}} + 2 - \sqrt{3} = \alpha - \frac{4}{\sqrt{3}}$$

$$\Rightarrow 2 - \frac{4}{\sqrt{3}} = \alpha - \frac{4}{\sqrt{3}} \Rightarrow \alpha = 2$$

26. Correct answer is [216].

Given, vertices of hyperbola = $(\pm 6, 0)$

Eccentricity, $e = \dfrac{\sqrt{5}}{2}$

As we know vertices of hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$ is $(\pm a, 0)$

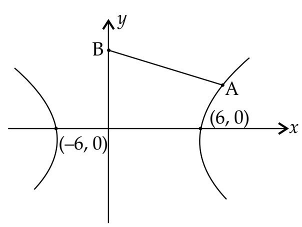
So, $a = 6$

As we know for $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1, e^2 = 1 + \dfrac{b^2}{a^2}$

$$\Rightarrow \frac{5}{4} = 1 + \frac{b^2}{36}$$

$$\Rightarrow b^2 = 9$$

So, H : $\dfrac{x^2}{36} - \dfrac{y^2}{9} = 1$



Let coordinates of A in parametric form be $(6 \sec \theta, 3 \tan \theta)$

So, slope of tangent at A = $\dfrac{3 \sec\theta}{6 \tan\theta} = \dfrac{1}{2\sin\theta}$

$\because$ Normal is parallel to the line $\sqrt{2}x + y = 2\sqrt{2}$

$$\therefore \frac{1}{2\sin\theta} \times (-\sqrt{2}) = -1$$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

{$\because$ A lies in first quadrant}

$$\therefore A = (6\sqrt{2}, 3)$$

Now, equation of normal is $\sqrt{2}x + y = k$

$\because$ Normal is passing through the $(6\sqrt{2}, 3)$

$$\Rightarrow \sqrt{2}(6\sqrt{2}) + 3 = k \Rightarrow k = 15$$

So, equation of normal is $\sqrt{2}x + y = 15$

So, B = (0, 15)

Now, d = AB

$$\Rightarrow d^2 = AB^2$$

$$\Rightarrow d^2 = \left[\sqrt{(6\sqrt{2})^2 + (12)^2}\right]^2$$

$$\Rightarrow d^2 = 72 + 144 \Rightarrow d^2 = 216$$

27. Correct answer is [120].

Given : x and y are distinct integers $1 \le x, y \le 25$

$\Rightarrow x + y$ must be multiple of 5.

$\Rightarrow x + y = 5k$ where $1 \le k \le 9$

Case 1: $x = 5k_2$, and $y = 5k_2$ where $k_1, k_2 \in \{1, 2, 3, 4, 5\}$

$\Rightarrow$ No. of ways = 5 × 4 = 20

Case 2: $x = 5k_1 + 1$ and $y = 5k_2 + 4$

$\Rightarrow$ No. of ways = 5 × 5 = 25

Case 3: $x = 5k_1 + 2$ and $y = 5k_2 + 3$

$\Rightarrow$ No. of ways = 5 × 5 = 25

Case 4: $x = 5k_1 + 3$ and $y = 5k_2 + 2$

$\Rightarrow$ No. of ways = 5 × 5 = 25

Case 5: $x = 5k_1 + 4$ and $y = 5k_2 + 1$

$\Rightarrow$ No. of ways = 5 × 5 = 25

$\Rightarrow$ Total no. of ways = 20 + 25 + 25 + 25 + 25 = 120

28. Correct answer is [25].

$$S = \{\alpha : \log_2 9^{2\alpha-4} + 13) - \log_2\left(\frac{5}{2}3^{2\alpha-4} + 1\right) = 2\}$$

$$\log_2\left\{\frac{(9^{2\alpha-4} + 13)}{\left(\frac{5}{2}3^{2\alpha-4} + 1\right)}\right\} = 2$$

$$\Rightarrow \frac{9^{2\alpha-4} + 13}{\frac{5}{2}3^{2\alpha-4} + 1} = 2^2$$

$\{\because \log_a b = c \Rightarrow a^c = b\}$

$$\Rightarrow 9^{2\alpha-4} + 13 = 4\left\{\frac{5}{2}3^{2\alpha-4} + 1\right\}$$

$$\Rightarrow 9^{2\alpha-4} + 13 = 10.3^{2\alpha-4} + 4$$

Let $3^{2\alpha-4} = k$

$\Rightarrow k^2 + 13 = 10k + 4$

$\Rightarrow k^2 - 10k + 9 = 0$

$\Rightarrow (k-9)(k-1) = 0 \Rightarrow k = 1, 9$

$\Rightarrow 3^{2\alpha-4} = 3^0$ and $3^{2\alpha-4} = 3^2$

$\Rightarrow 2\alpha - 4 = 0$ and $2\alpha - 4 = 2$

$\Rightarrow \alpha = 2, 3$

$$x^2 - 2\left(\sum_{\alpha \in S} \alpha\right)^2 x + \sum_{\alpha \in S} (\alpha+1)^2 \beta = 0$$

$\Rightarrow x^2 - 2(2+3)^2 x + (3^2 + 4^2)\beta = 0$

$\Rightarrow x^2 - 50x + 25\beta = 0$

The equation has real roots when discriminant ≥ 0.

Discriminant of $ax^2 + bx + c = 0$ is $b^2 - 4ac$

$\Rightarrow (-50)^2 - 4(25\beta) \ge 0$

$\Rightarrow 2500 - 100\beta \ge 0$

$\Rightarrow \beta \le 25 \Rightarrow \beta_{max} = 25$

29. Correct answer is [600].

Given $P_1 : 2y = 5x^2$ and $P_2 : x^2 - y + 6 = 0$

And $y = \alpha x$

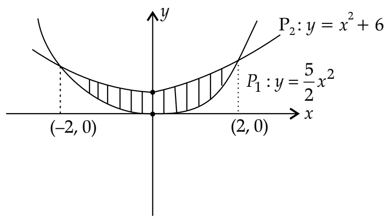
So, $P_1 : y = \frac{5}{2}x^2$ and $P_2 : y = x^2 + 6$

Lets find intersecting points of P_1 and P_2

$\frac{5}{2}x^2 = x^2 + 6$

$\Rightarrow 3x^2 = 12 \Rightarrow x = \pm 2$

$\Rightarrow$ Intersecting points are (2, 10) and (–2, 10)



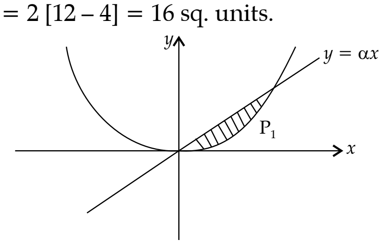
Let A_1 = Area enclosed by parabola P_1 and P_2

$\Rightarrow A_1 = 2\int_0^2 \left(x^2 + 6 - \frac{5}{2}x^2\right)dx$

$\Rightarrow A_1 = 2\int_0^2 \left(6 - \frac{3}{2}x^2\right)dx$

$\Rightarrow A_1 = 2\left[6x - \frac{1}{2}x^3\right]_0^2$

$\Rightarrow A_1 = 2\,[12 - 4] = 16$ sq. units.



Lets find intersecting points of $y = \alpha x$ and P_1

$\alpha x = \frac{5}{2}x^2 \Rightarrow x\left(\alpha - \frac{5}{2}x\right) = 0 \Rightarrow x = 0, \frac{2\alpha}{5}$

So, intersecting points are (0, 0) and $\left(\frac{2\alpha}{5}, \frac{2\alpha^2}{5}\right)$

Let A_2 = Area enclosed by P_1 and $y = \alpha x$

$\Rightarrow A_2 = \frac{1}{2}\left(\frac{2\alpha}{5}\right)\left(\frac{2\alpha^2}{5}\right) - \int_0^{\frac{2\alpha}{5}} \frac{5x^2}{2}dx$

$\Rightarrow A_2 = \frac{2\alpha^3}{25} - \frac{5}{2}\left[\frac{x^3}{3}\right]_0^{\frac{2\alpha}{5}}$

$\Rightarrow A_2 = \frac{2\alpha^3}{25} - \frac{5}{6}\left[\frac{2\alpha}{5}\right]^3$

$\Rightarrow A_2 = \frac{2\alpha^3}{25} - \frac{20}{3}\left(\frac{\alpha^3}{125}\right) \Rightarrow A_2 = \frac{2\alpha^3}{75}$

$\because A_1 = A_2$

$\Rightarrow \frac{2\alpha^3}{75} = 16 \Rightarrow \alpha^3 = 8 \times 75 \Rightarrow \alpha^3 = 600$

HINT:

Area enclosed by two curve $y_1 = f(x)$, $y_2 = g(x)$ and line $x = 0$, $x = b$ $\{b > a\}$ is given by

$A = \left|\int_a^b [f(x) - g(x)]dx\right|$

30. Correct answer is [495].

A_1, A_2, A_3 are three A.P.s. with common difference d and first terms are A, A + 1, A + 2

As we know, n^{th} term of A.P. $= a + (n-1)d$.

$\Rightarrow a = A + 6d$, $b = (A + 1) + 8d$

and $c = (A + 2) + 16d$

$\therefore \begin{vmatrix} a & 7 & 1 \\ 2b & 17 & 1 \\ c & 17 & 1 \end{vmatrix} + 70 = 0$

$\Rightarrow \begin{vmatrix} A+6d & 7 & 1 \\ 2(A+1+8d) & 17 & 1 \\ A+2+16d & 17 & 1 \end{vmatrix} + 70 = 0$

Applying $R_3 \to R_3 - R_2$, we get

$\Rightarrow \begin{vmatrix} A+6d & 7 & 1 \\ 2A+2+16d & 17 & 1 \\ -A & 0 & 0 \end{vmatrix} + 70 = 0$

$\Rightarrow -A\,(7 - 17) + 70 = 0$

$\Rightarrow 10A + 70 = 0$

$\Rightarrow 10A = -70 \Rightarrow A = -7$

$\because a = A + 6d$

$\Rightarrow 29 = -7 + 6d$

$\Rightarrow 6d = 36 \Rightarrow d = 6$

$\therefore b = (-7 + 1) + 8(6)$

$\Rightarrow b = 42$ and $c = (-7 + 2) + 16\,(6)$

$\Rightarrow c = 91$

So, first term of A.P. $= c - a - b$

$= 91 - 29 - 42 = 20$

and common difference $= \frac{d}{12} = \frac{1}{2}$

As we know, sum of n terms of AP with common difference d and first term a is $\frac{n}{2}\{2a + (n-1)d\}$.

$\Rightarrow S_{20} = \frac{20}{2}\left\{2(20) + 19\left(\frac{1}{2}\right)\right\}$

$\Rightarrow S_{20} = 10\,\{40 + 9.5\}$

$\Rightarrow S_{20} = 495$

HINT:

Use n^{th} terms of an A.P. with first term a and common difference d is $a + (n-1)\,d$ and sum of n terms of AP is $\frac{n}{2}\{2a + (n-1)d\}$.

❑❑