JEE (Main) MATHEMATICS SOLVED PAPER

2023
25th Jan. Shift 2

Section A

Q. 1. Let $\Delta, \nabla \in (\wedge \vee)$ be such that $(p \to q)\Delta(p \nabla q)$ is a tautology. Then

(1) $\Delta = \vee, \nabla = \vee$ (2) $\Delta = \vee, \nabla = \wedge$

(3) $\Delta = \wedge, \nabla = \vee$ (4) $\Delta = \wedge, \nabla = \wedge$

Q. 2. If the four points, whose position vectors are $3\hat{i} - 4\hat{j} + 2\hat{k}, \hat{i} + 2\hat{j} - \hat{k}, -2\hat{i} - \hat{j} + 3\hat{k}$, and $5\hat{i} - 2\alpha\hat{j} + 4\hat{k}$ are coplanar, then α is equal to

(1) $\frac{73}{17}$ (2) $\frac{107}{17}$ (3) $\frac{-73}{17}$ (4) $\frac{-107}{17}$

Q. 3. The foot of perpendicular of the point (2, 0, 5) on the line $\frac{x+1}{2} = \frac{y-1}{5} = \frac{z+1}{-1}$ is (α, β, γ). Then which of the following is NOT correct?

(1) $\frac{\beta}{\gamma} = -5$ (2) $\frac{\gamma}{\alpha} = \frac{5}{8}$

(3) $\frac{\alpha}{\beta} = -8$ (4) $\frac{\alpha\beta}{\gamma} = \frac{4}{15}$

Q. 4. The equations of two sides of a variable triangle are $x = 0$ and $y = 3$, and its third side is a tangent to parabola $y^2 = 6x$. The locus of its circumcentre is:

(1) $4y^2 - 18y - 3x - 18 = 0$
(2) $4y^2 - 18y - 3x + 18 = 0$
(3) $4y^2 - 18y + 3x + 18 = 0$
(4) $4y^2 + 18y + 3x + 18 = 0$

Q. 5. Let $f(x) = 2x^n + \lambda, \lambda \in \mathbb{R}, n \in \mathrm{N}$, and $f(4) = 133, f(5) = 255$. Then the sum of all the positive integer divisors of $(f(3) - f(2))$ is

(1) 60 (2) 59 (3) 61 (4) 58

Q. 6. $\sum_{K=0}^{6} {}^{51-k}C_3$ is equal to

(1) ${}^{51}C_4 - {}^{45}C_4$ (2) ${}^{52}C_3 - {}^{45}C_3$
(3) ${}^{52}C_4 - {}^{45}C_4$ (4) ${}^{51}C_3 - {}^{45}C_3$

Q. 7. Let the function $f(x) = 2x^3 + (2p - 7)x^2 + 3(2p - 9)x - 6$ have a maxima for some value of $x < 0$ and a minima for some value of $x > 0$.Then, the set of all values of p is

(1) $\left(0, \frac{9}{2}\right)$ (2) $\left(-\infty, \frac{9}{2}\right)$

(3) $\left(-\frac{9}{2}, \frac{9}{2}\right)$ (4) $\left(\frac{9}{2}, \infty\right)$

Q. 8. Let $A = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}$, where $i = \sqrt{-1}$.

If $M = A^T BA$, then the inverse of the matrix $AM^{2023}A^T$ is

(1) $\begin{bmatrix} 1 & 0 \\ -2023i & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & -2023i \\ 0 & 1 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & 0 \\ 2023i & 1 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & 2023i \\ 0 & 1 \end{bmatrix}$

Q. 9. Let $\vec{a} = -\hat{i} - \hat{j} + \hat{k}, \vec{a}.\vec{b} = 1$ and $\vec{a} \times \vec{b} = \hat{i} - \hat{j}$. Then $\vec{a} - 6\vec{b}$ is equal to

(1) $3(\hat{i} - \hat{j} + \hat{k})$ (2) $(\hat{i} + \hat{j} - \hat{k})$

(3) $3(\hat{i} + \hat{j} + \hat{k})$ (4) $3(\hat{i} - \hat{j} - \hat{k})$

Q. 10. The integral $16\int_1^2 \frac{dx}{x^3(x^2+2)^2}$ is equal to

(1) $\frac{11}{12} - \log_e 4$ (2) $\frac{11}{6} - \log_e 4$

(3) $\frac{11}{6} + \log_e 4$ (4) $\frac{11}{12} + \log_e 4$

Q. 11. Let T and C respectively be the transverse and conjugate axes of the hyperbola $16x^2 - y^2 + 64x + 4y + 44 = 0$. Then the area of the region above the parabola $x^2 = y + 4$, below the transverse axis T and on the right of the conjugate axis C is:

(1) $4\sqrt{6} + \frac{28}{3}$ (2) $4\sqrt{6} - \frac{44}{3}$

(3) $4\sqrt{6} + \frac{44}{3}$ (4) $4\sqrt{6} - \frac{28}{3}$

Q. 12. Let N be the sum of the numbers appeared when two fair dice are rolled and let the probability that $N - 2, \sqrt{3N}, N + 2$ are in geometric progression be $\frac{k}{48}$. Then the value of k is

(1) 8 (2) 16 (3) 2 (4) 4

Q. 13. If the function $f(x) =$

$$\begin{cases} (1+|\cos x|)^{\frac{\lambda}{|\cos x|}}, & 0 < x < \frac{\pi}{2} \\ \mu & x = \frac{\pi}{2} \\ \frac{\cot 6x}{e^{\cot 4x}} & \frac{\pi}{2} < x < \pi \end{cases}$$

is continuous at $x = \frac{\pi}{2}$, then $9\lambda + 6\log_e \mu + \mu^6 - e^{6\lambda}$ is equal to

(1) 10 **(2)** $2e^4 + 8$ **(3)** 11 **(4)** 8

Q. 14. The number of functions $f : \{1, 2, 3, 4\} \rightarrow \{a \in Z : |a| \le 8\}$ satisfying $f(n) + \frac{1}{n}f(n+1) = 1$, $\forall n \in \{1,2,3\}$ is

(1) 1 **(2)** 4 **(3)** 2 **(4)** 3

Q. 15. Let $y = y(t)$ be a solution of the differential equation $\frac{dy}{dt} + \alpha y = \gamma e^{-\beta t}$ where, $\alpha > 0$, $\beta > 0$ and $\gamma > 0$. Then $\lim_{t\to\infty} y(t)$

(1) is –1 **(2)** is 1
(3) does not exist **(4)** is 0

Q. 16. Let z be a complex number such that $\left|\frac{z-2i}{z+1}\right| = 2, z \ne i$. Then z lies on the circle of radius 2 and centre

(1) (2, 0) **(2)** (0, 2) **(3)** (0, –2) **(4)** (0, 0)

Q. 17. Let A, B, C be 3 × 3 matrices such that A is symmetric and B and C are skew-symmetric. Consider the statements

(S1) $A^{13} B^{26} - B^{26} A^{13}$ is symmetric

(S2) $A^{26}C^{13} - C^{13} A^{26}$ is symmetric

Then,

(1) Only S2 is true
(2) Both S1 and S2 are false
(3) Only S1 is true
(4) Both S1 and S2 are true

Q. 18. The number of numbers, strictly between 5000 and 10000 can be formed using the digits 1,3,5,7,9 without repetition, is

(1) 12 **(2)** 120 **(3)** 72 **(4)** 6

Q. 19. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by

$f(x) = \log_{\sqrt{m}}\{\sqrt{2}(\sin x - \cos x) + m - 2\}$ for some m, such that the range of f is [0,2]. Then the value of m is

(1) 5 **(2)** 4 **(3)** 3 **(4)** 2

Q. 20. The shortest distance between the lines $x + 1 = 2y = -12z$ and $x = y + 2 = 6z - 6$ is

(1) $\frac{3}{2}$ **(2)** 2 **(3)** $\frac{5}{2}$ **(4)** 3

Section B

Q. 21. 25% of the population are smokers. A smoker has 27 times more chances to develop lung cancer than a non smoker. A person is diagnosed with lung cancer and the probability that this person is a smoker is $\frac{k}{10}$. Then the value of k is.

Q. 22. The remainder when $(2023)^{2023}$ is divided by 35 is

Q. 23. Let $a \in \mathbb{R}$ and let α, β be the roots of the equation

$x^2 + 60^{\frac{1}{4}}x + a = 0$

If $\alpha^4 + \beta^4 = -30$, then the product of all possible values of a is

Q. 24. For the two positive numbers a, b is a, b and $\frac{1}{18}$ are in a geometric progression, while $\frac{1}{a}$, 10 and $\frac{1}{b}$ are in an arithmetic progression, then $16a + 12b$ is equal to

Q. 25. If m and n respectively are the numbers of positive and negative values of q in the interval $[-\pi, \pi]$ that satisfy the equation $\cos 2\theta \cos\frac{\theta}{2} = \cos 3\theta \cos\frac{9\theta}{2}$, , then mn is equal to

Q. 26. If the shortest distance between the line joining the points (1, 2, 3) and (2, 3, 4), and the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-2}{0}$ is a. then $28a^2$ is equal to

Q. 27. Points P(–3, 2), Q(9, 10) and R(a, 4) lie on a circle C with PR as its diameter, The tangents to C at the points Q and R intersect at the point S. If S lies on the line $2x - ky = 1$, then k is equal to

Q. 28. Suppose Anil's mother wants to give 5 whole fruits to Anil from a basket of 7 red apples, 5 white apples and 8 oranges. If in the selected 5 fruits, at least 2 oranges, at least one red apple and at least one white apple must be given, then the number of ways, Anil's mother can offer 5 fruits to Anil is

Q. 29. If $\int_{\frac{1}{3}}^{3} |\log_e x|\, dx = \frac{m}{n}\log_e\left(\frac{n^2}{e}\right)$, where m and n are coprime natural numbers, then $m^2 + n^2 - 5$ is equal to

Q. 30. A triangle is formed by X-axis, Y-axis and the line $3x + 4y = 60$. Then the number of points P(a, b) which lie strictly inside the triangle, where a is an integer and b is a multiple of a, is

Answer Key

Q. No.	Answer	Topic Name	Chapter Name
1	(1)	Tautology and Contradiction	Mathematical Reasoning
2	(1)	Triple Products	Vector Algebra
3	(1)	Foot of perpendicular	Three Dimensional Geometry
4	(3)	Tangent to a Parabola	Conic Section
5	(1)	Algebra of Functions	Functions
6	(3)	Properties of Binomial Coefficients	Binomial Theorem
7	(2)	Maxima and Minima	Application of Derivatives
8	(4)	Inverse of a Matrix	Matrices and Determinants
9	(3)	Scalar and Vector Products	Vector Algebra
10	(2)	Basics of Definite Integration	Integral Calculus
11	(1)	Area Bounded by Curves	Area under Curves
12	(4)	Basics of Probability	Probability
13	(1)	Continuity of a Function	Continuity and Differentiability
14	(3)	Basics of Functions	Function
15	(4)	Linear Differential Equations	Differential Equations
16	(3)	Algebra of Complex Numbers	Complex Numbers
17	(1)	Symmetric and Skew-Symmetric Matrices	Matrices and Determinants
18	(3)	Permutations	Permutations and Combinations
19	(1)	Algebra of Functions	Functions
20	(2)	Shortest Distance	Three Dimensional Geometry
21	[9]	Bayes' Theorem	Probability
22	[7]	Binomial Theorem for Positive Integral Index	Binomial Theorem
23	[45]	Relation between Roots and Coefficients	Quadratic Equations
24	[3]	Geometric Progressions	Sequences and Series
25	[25]	Trigonometric Equations	Trigonometric Equations and Inequalities
26	[18]	Shortest Distance	Three Dimensional Geometry
27	[3]	Tangent and Normal of a Circle	Circle
28	[6860]	Combinations	Permutations and Combinations
29	[20]	Properties of Definite Integrals	Integral Calculus
30	[31]	Interaction between Two Lines	Point and Straight Line

Solutions

Section A

1. **Option (1) is correct.**

(A) $(p \to q)\, \Delta\, (p \,\nabla\, q) = (\sim p \vee q)\, \Delta\, (p \,\nabla\, q)$

$\{\because A \to B = \sim A \vee B\}$

If $\Delta = \vee$, $\nabla = \vee$, then

$(p \to q)\, \Delta\, (p \,\nabla\, q) = (\sim p \vee q) \vee (p \vee q)$

$= \sim p \vee p \vee q = T$

(B) If $\Delta = \vee$ $\nabla = \wedge$, then

$(p \to q)\, \Delta\, (p \,\nabla\, q) = (\sim p \vee q) \vee (p \wedge q)$

$= (\sim p \vee q \vee p) \wedge (\sim p \vee q \vee q)$

$\{\because A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)\}$

$= T \wedge \{\sim p \vee q) = \sim p \vee q$

(C) If $\Delta = \wedge$, $\nabla = \vee$, then

$(p \to q)\, \Delta\, (p \,\nabla\, q) = (\sim p \vee q) \wedge (p \vee q)$

$= [(\sim p \vee q) \wedge p] \vee [(\sim p \vee q) \wedge q]$

$= \{(p \wedge \sim p) \vee (p \wedge q)\} \vee \{(\sim p \vee q) \wedge q\}$

$= \{F \vee (p \wedge q)\} \vee \{(\sim p \vee q) \wedge q\} = q$

(D) If $\Delta = \wedge$, $\nabla = \wedge$, then

$(p \to q)\, \Delta\, (p \,\nabla\, q) = (\sim p \vee q) \wedge (p \wedge q)$

$= [(\sim p \vee q) \wedge p] \wedge q$

$= [F \vee (p \wedge q)] \wedge q$

$= p \wedge q \wedge q = p \wedge q$

2. Option (1) is correct.

Let $\vec{A} = 3\hat{i} - 4\hat{j} + 2\hat{k}$

$\vec{B} = \hat{i} + 2\hat{j} - \hat{k}$

$\vec{C} = -2\hat{i} - \hat{j} + 3\hat{k}$

$\vec{D} = 5\hat{i} - 2\alpha\hat{j} + 4\hat{k}$

Now, $\overrightarrow{AB} = -2\hat{i} + 6\hat{j} - 3\hat{k}$

$\overrightarrow{BC} = -3\hat{i} - 3\hat{j} + 4\hat{k}$

$\overrightarrow{CD} = 7\hat{i} + (1 - 2\alpha)\hat{j} + \hat{k}$

$\because$ Given points are coplanar

$\therefore [\overrightarrow{AB}\ \overrightarrow{BC}\ \overrightarrow{CD}] = 0$

$$\Rightarrow \begin{vmatrix} -2 & 6 & -3 \\ -3 & -3 & 4 \\ 7 & 1-2\alpha & 1 \end{vmatrix} = 0$$

$\Rightarrow -2(-3 - 4 + 8\alpha) + 3(6 + 3 - 6\alpha) + 7(24 - 9) = 0$

$\Rightarrow 14 - 16\alpha + 27 - 18\alpha + 105 = 0$

$\Rightarrow -34\alpha + 146 = 0$

$\Rightarrow \alpha = \dfrac{73}{17}$

HINT:

If four points A, B, C, D are coplanar, then

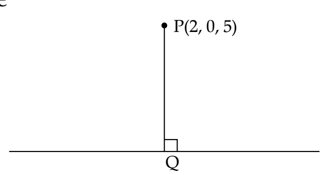
$$[\overrightarrow{AB}\ \overrightarrow{BC}\ \overrightarrow{CD}] = 0$$

3. Option (1) is correct.

Given, equation of line

is $\dfrac{x+1}{2} = \dfrac{y-1}{5} = \dfrac{z+1}{-1}$

And point P = (2, 0, 5)



Let $\dfrac{x+1}{2} = \dfrac{y-1}{5} = \dfrac{z+1}{-1} = k$

Let coordinates of point Q be $(2k - 1, 5k + 1, -k-1)$

Now, direction ratios of PQ = $(2k - 3, 5k + 1, -k-6)$

$\because$ PQ $\perp$ line

$\therefore (2k - 3)\,2 + (5k + 1)\,5 + (-k - 6)(-1) = 0$

$\Rightarrow 4k - 6 + 25k + 5 + k + 6 = 0$

$\Rightarrow 30k = -5$

$\Rightarrow k = -\dfrac{1}{6}$

So, coordinates of point

$Q = \left(2\left(\dfrac{-1}{6}\right) - 1, 5\left(\dfrac{-1}{6}\right) + 1, \dfrac{1}{6} - 1\right)$

$= \left(\dfrac{-4}{3}, \dfrac{1}{6}, \dfrac{-5}{6}\right)$

$\therefore \alpha = \dfrac{-4}{3}, \beta = \dfrac{1}{6}, \gamma = \dfrac{-5}{6}$

(A) $\dfrac{\beta}{\gamma} = \dfrac{-1}{5}$

So, $\dfrac{\beta}{\gamma} \neq -5$

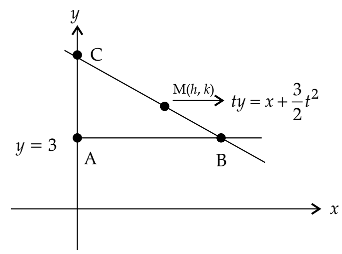
(B) $\dfrac{\gamma}{\alpha} = \dfrac{-5/6}{-4/3} = \dfrac{5}{8}$

(C) $\dfrac{\alpha}{\beta} = \dfrac{-4/3}{1/6} = -8$

(D) $\dfrac{\alpha\beta}{\gamma} = \left(-\dfrac{4}{3} \times \dfrac{1}{6}\right) \times \left(\dfrac{6}{-5}\right) = \dfrac{4}{15}$

4. Option (3) is correct.

Equation of two sides of variable triangle are $x = 0$ and $y = 3$.



Now, equation of tangent to parabola $y^2 = 6x$ in parametric form is given by $ty = x + \dfrac{3}{2}t^2$ ...(i)

On solving equation (i) with $y = 3$, we get

$B = \left(\dfrac{6t - 3t^2}{2}, 3\right)$

On solving equation (i) with $x = 0$, we get

$C = \left(0, \dfrac{3t}{2}\right)$

Let coordinates of circumcentre M = (h, k)

As we know in right angled triangle, circumcentre will be the midpoint of hypotenuse.

So, $(h, k) = \left(\dfrac{6t - 3t^2}{4}, \dfrac{3 + \dfrac{3t}{2}}{2}\right)$

$\Rightarrow (h, k) = \left(\dfrac{6t - 3t^2}{4}, \dfrac{6 + 3t}{4}\right)$

$\Rightarrow h = \dfrac{6t - 3t^2}{4}$ ...(ii)

and $k = \dfrac{6 + 3t}{4}$ ...(iii)

From equation (iii), $t = \dfrac{4k - 6}{3}$

Put value of t in equation (ii), we get

$4h = 6\left(\dfrac{4k - 6}{3}\right) - 3\left(\dfrac{4k - 6}{3}\right)^2$

$\Rightarrow 4h = 8k - 12 - \dfrac{1}{3}(16k^2 + 36 - 48k)$

$\Rightarrow 12h - 24k + 36 = -(16k^2 - 48k + 36)$

$\Rightarrow 16k^2 - 72k + 12h + 72 = 0$

$\Rightarrow 4k^2 - 18k + 3h + 18 = 0$

$\therefore$ Locus will be $4y^2 - 18y + 3x + 18 = 0$

5. Option (1) is correct.

Given $f(x) = 2x^n + \lambda, \lambda \in R$

And $f(4) = 133$

And $f(5) = 255$

$\because f(4) = 133$

$\Rightarrow 2(4)^n + \lambda = 133$ ...(i)

And $f(5) = 255$

$\Rightarrow 2(5)^n + \lambda = 255$ ...(ii)

Equation (ii) – equation (i), we get

$2(5)^n - 2(4)^n = 122$

$\Rightarrow 5^n - 4^n = 61 \Rightarrow n = 3$

Now, $f(3) = 2(3)^3 + \lambda = 54 + \lambda$

$f(2) = 2(2)^3 + \lambda = 16 + \lambda$

So, $f(3) - f(2) = 38 = 2 \times 19$

Now, integer divisors of $f(3) - f(2) = 1, 2, 19, 38$

So, sum of all the positive integer divisors

$= 1 + 2 + 19 + 38 = 60$

HINT:

Find the value of n by using given conditions and then find $f(3) - f(2)$ and solve further.

6. Option (3) is correct.

Let $A = \sum_{k=0}^{6} {}^{51-k}C_3$

$\Rightarrow A = {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3 + {}^{46}C_3 + {}^{45}C_3$

Add and subtract ${}^{45}C_4$, we get

$A = ({}^{45}C_4 + {}^{45}C_3) + {}^{46}C_3 + {}^{47}C_3 + ... + {}^{51}C_3 - {}^{45}C_4$

As we know ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

$\Rightarrow A = {}^{46}C_4 + {}^{46}C_3 + {}^{47}C_3 + ... + {}^{51}C_3 - {}^{45}C_4$

$A = {}^{47}C_4 + {}^{47}C_3 + ... + {}^{51}C_3 - {}^{45}C_4$

$= {}^{48}C_4 + {}^{48}C_3 + ... + {}^{51}C_3 - {}^{45}C_4$

$= {}^{49}C_4 + {}^{49}C_3 + ... + {}^{51}C_3 - {}^{45}C_4$

$= {}^{50}C_4 + {}^{50}C_3 + ... + {}^{51}C_3 - {}^{45}C_4$

$= {}^{51}C_4 + {}^{51}C_3 - {}^{45}C_3$

$\Rightarrow A = {}^{52}C_4 - {}^{45}C_4$

7. Option (2) is correct.

Given: $(f(x) = 2x^3 + (2p - 7)x^2 + 3(2p - 9)x - 6$

$\Rightarrow f'(x) = 6x^2 + 2x(2p - 7) + 3(2p - 9)$

Let roots of $f'(x) = 0$ be x_1 and x_2

$\because f(x)$ have a maxima for some value of $x < 0$ and a minima for some value of $x > 0$.

$\therefore x_1 x_1 < 0$

$\Rightarrow \frac{3(2p-9)}{6} < 0 \Rightarrow p - \frac{9}{2} < 0$

$\Rightarrow p < \frac{9}{2}$

HINT:

(1) Use function have maxima and minima at critical points.

(2) For critical points $f'(x) = 0$

8. Option (4) is correct.

Given : $A = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}, B = \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}$

And $M = A^TBA$

Now, $M^2 = (A^TBA)(A^TBA)$

$\Rightarrow M^2 = A^TBI\,B\,A$ $\{\because AA^T = I]$

$\Rightarrow M^2 = A^TB^2A$

Similarly $M^3 = (A^TB^2A)(A^TBA)$

$\Rightarrow M^3 = A^TB^3A$

$\therefore M^{2023} = A^TB^{2023}A$

Let $P = AM^{2023}A^T$

$\Rightarrow P = AA^T B^{2023} AA^T$

$\Rightarrow P = B^{2023}$ $\{\because AA^T = A^TA = I\}$

Now, $B^2 = \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}$

$\Rightarrow B^2 = \begin{bmatrix} 1 & -2i \\ 0 & 1 \end{bmatrix}$

Similarly, $B^3 = \begin{bmatrix} 1 & -2i \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -3i \\ 0 & 1 \end{bmatrix}$

$\therefore B^{2023} = \begin{bmatrix} 1 & -2023i \\ 0 & 1 \end{bmatrix}$

Now, $P^{-1} = \begin{bmatrix} 1 & 2023i \\ 0 & 1 \end{bmatrix}$

9. Option (3) is correct.

Given, $\vec{a} = -\hat{i} - \hat{j} + \hat{k}$

$\vec{a}.\vec{b} = 1$

$\vec{a} \times \vec{b} = \hat{i} - \hat{j}$

Let $\vec{b} = p\hat{i} + q\hat{j} + r\hat{k}$

$\because \vec{a}.\vec{b} = 1$

$\Rightarrow (-\hat{i} - \hat{j} + \hat{k}).(p\hat{i} + q\hat{j} + r\hat{k}) = 1$

$\Rightarrow -p - q + r = 1$ ...(i)

Also, $\vec{a} \times \vec{b} = \hat{i} - \hat{j}$

$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 1 \\ p & q & r \end{vmatrix} = \hat{i} - \hat{j}$

$\Rightarrow -(r+q)\hat{i} + (p+r)\hat{j} + (p-q)\hat{k} = \hat{i} - \hat{j}$

$\Rightarrow -(r + q) = 1$ ......(ii) $(p + r) = -1$ ....(iii)

and $p - q = 0$ ...(iv)

On solving equation (i), (ii), (iii) and (iv), we get

$p = -\frac{2}{3}, q = -\frac{2}{3}, r = -\frac{1}{3}$

$\therefore \vec{b} = \frac{-2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$

$\Rightarrow 6\vec{b} = -4\hat{i} - 4\hat{j} - 2\hat{k}$

Now, $\vec{a} - 6\vec{b} = (-\hat{i} - \hat{j} + \hat{k}) - (-u\hat{i} - u\hat{j} - 2\hat{k})$

$= 3\hat{i} + 3\hat{j} + 3\hat{k} = 3(\hat{i} + \hat{j} + \hat{k})$

10. Option (2) is correct.

Let $I = 16\int_1^2 \frac{dx}{x^3(x^2+2)^2}$

$\Rightarrow I = 16\int_1^2 \frac{dx}{x^3 x^4\left(1+\frac{2}{x^2}\right)^2}$

Let $1+\frac{2}{x^2} = y$

$\Rightarrow \frac{-4}{x^3}dx = dy \Rightarrow \frac{dx}{x^3} = \frac{-dy}{4}$ and $1+\frac{2}{x^2} = y$

So, $x = 1 \Rightarrow y = 3$

and $x = 2 \Rightarrow y = \frac{3}{2}$

$\Rightarrow \frac{2}{x^2} = y - 1 \Rightarrow x^2 = \frac{2}{y-1}$

$\Rightarrow I = \frac{-16}{4}\int_3^{\frac{3}{2}} \frac{(y-1)^2}{4y^2}dy$

$\Rightarrow I = \int_3^{\frac{3}{2}} \frac{y^2+1-2y}{y^2}dy$

$\Rightarrow I = -\int_{\frac{3}{2}}^3 \left(1+\frac{1}{y^2}-\frac{2}{y}\right)dy$

$\Rightarrow I = \left[y - \frac{1}{y} - 2\ln y\right]_{\frac{3}{2}}^3$

$\Rightarrow I = 3 - \frac{3}{2} - \frac{1}{3} + \frac{2}{3} - 2\left(\ln 3 - \ln\left(\frac{3}{2}\right)\right)$

$\Rightarrow I = \frac{11}{6} - 2\ln 3 + 2\ln 3 - 2\ln 2$

$\Rightarrow I = \frac{11}{6} - 2\ln 2 \Rightarrow I = \frac{11}{6} - \ln 4$

HINT:

Assume $1+\frac{2}{x^2} = y$ and use integration by substitution.

11. Option (1) is correct.

Given: Equation of hyperbola is

$16x^2 - y^2 + 64x + 4y + 44 = 0$

$\Rightarrow 16(x^2 + 4x) - (y^2 - 4y) + 44 = 0$

$\Rightarrow 16[(x+2)^2 - 4] - [(y-2)^2] + 4 + 44 = 0$

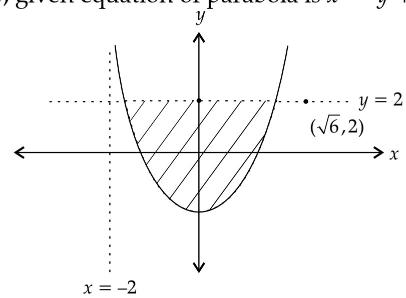
$\Rightarrow 16(x+2)^2 - (y-2)^2 = 16$

$\Rightarrow \frac{(x+2)^2}{1} - \frac{(y-2)^2}{4^2} = 1$

$\therefore$ Transverse axis T : $y = 2$

And conjugate axis C : $x = -2$

Now, given equation of parabola is $x^2 = y + 4$



Now, required area $A = \int_{-2}^{\sqrt{6}} [2-(x^2-4)]dx$

$\Rightarrow A = \left[6x - \frac{x^3}{3}\right]_{-2}^{\sqrt{6}}$

$\Rightarrow A = \left[6\sqrt{6} - \frac{6\sqrt{6}}{3}\right] - \left[-12 + \frac{8}{3}\right]$

$\Rightarrow A = \frac{12\sqrt{6}}{3} + \frac{28}{3} = 4\sqrt{6} + \frac{28}{3}$

12. Option (4) is correct.

Given: N = sum of the number appeared when two fair dice are rolled.

And $N-2, \sqrt{3N}, N+2$ are in G.P.

So, $(\sqrt{3N})^2 = (N-2)(N+2)$

$\Rightarrow 3N = N^2 - 4$

$\Rightarrow N^2 - 3N - 4 = 0$

$\Rightarrow (N-4)(N+1) = 0$

$\Rightarrow N = 4, -1$ (Not possible)

So, sum of the number appeared when two fair dice are rolled = 4

Now, possible outcomes = (1, 3), (2, 2), (3, 1)

And total outcomes when two dice are rolled = 36

$\therefore$ Required probability $= \frac{3}{36} = \frac{k}{48}$

$\Rightarrow k = 4$

13. Option (1) is correct.

Given, $f(x) = \begin{cases} (1+|\cos x|)^{\frac{\lambda}{|\cos x|}} & ; \quad 0 < x < \frac{\pi}{2} \\ \mu & ; \quad x = \frac{\pi}{2} \\ e^{\frac{\cot 6x}{\cot 4x}} & ; \quad \frac{\pi}{2} < x < \pi \end{cases}$

$\because f(x)$ is continuous at $x = \frac{\pi}{2}$

$\Rightarrow \lim_{x\to\left(\frac{\pi}{2}\right)^-} f(x) = \lim_{x\to\left(\frac{\pi}{2}\right)^+} f(x) = f\left(\frac{\pi}{2}\right)$

Now, $\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^+} (1+|\cos x|)^{\left(\frac{\lambda}{|\cos x|}\right)}$

$\because$ Above limit has 1^{∞} indeterminate form.

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = e^{\lim_{x\to\frac{\pi}{2}^-}[1+|\cos x|-1]\frac{\lambda}{|\cos x|}}$$

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = e^{\lim_{x\to\frac{\pi}{2}^-}\lambda e^{\lambda}}$$

$$\text{Now, } \lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^+} e^{\frac{\cot 6x}{\cot 4x}}$$

Let $A = \dfrac{\cot 6x}{\cot 4x}$

So, $\lim_{x\to\frac{\pi}{2}^+} A = \lim_{x\to\frac{\pi}{2}^+} \dfrac{\tan 4x}{\tan 6x}\left(\dfrac{0}{0}\text{ form}\right)$

$$= \lim_{x\to\frac{\pi}{2}^+} \frac{(\sec^2 4x)4}{(\sec^2 6x)6} = \frac{2}{3}$$

So, $\lim_{x\to\frac{\pi}{2}^+} f(x) = e^{\frac{2}{3}}$

$\therefore e^{\lambda} = e^{\frac{2}{3}} = \mu$

$\Rightarrow \lambda = \dfrac{2}{3}$ and $\mu = e^{\frac{2}{3}}$

Now, $9\lambda + 6\log_e \mu + \mu^6 - e^{6\lambda}$

$$= 9\left(\frac{2}{3}\right) + 6\log_e e^{\frac{2}{3}} + \left(e^{\frac{2}{3}}\right)^6 - e^{6\left(\frac{2}{3}\right)}$$

$$= 6 + 6\times\frac{2}{3} = 6 + 4 = 10$$

14. Option (3) is correct.

Given : $f: \{1, 2, 3, 4\} \to \{a \in Z : |a| < 8\}$

Also, $f(n) + \dfrac{1}{n} f(n+1) = 1 \qquad n \in \{1,2,3\}$

$\Rightarrow nf(n) + f(n + 1) = n$

$\Rightarrow f(n + 1) = n - n\, f(n)$

$\Rightarrow f(n + 1) = n\,(1 - f(n))$

For $n = 1$, $f(2) = (1 - f(1))$

For $n = 2$, $f(3) = 2\,(1 - f(2))$

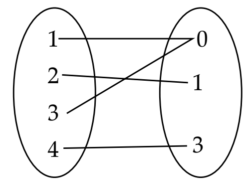
$= 2\,(1 - 1 + f(1)) = 2f(1)$

For $n = 3$, $f(4) = 3\,(1 - f(3))$

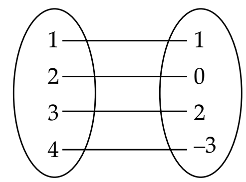
$= 3\,(1 - 2f(1)) = 3 - 6f(1)$

$\therefore f(1) = f(1), f(2) = 1 - f(1), f(3) = 2\,f(1), f(4) = 3 - 6\,f(1)$

Now, for $f(1) = 0, f(2) = 1, f(3) = 0, f(4) = 3$



For $f(1) = 1, f(2) = 0, f(3) = 2, f(4) = -3$



For $f(1) = -1, f(2) = 2, f(3) = -2, f(4) = 9$

$\because f(4) = 9$ is not in the range of function $f, |a| < 8$

$\therefore$ This is not a function.

Similarly for $f(1) = \{-7, -6, -5, -4, -3, -2, -1, 2, 3, 4, 5, 6, 7\}$, f is not a function.

$\therefore$ Only two functions are possible.

HINT:

(1) Find $f(n + 1)$ in terms of $f(n)$ and find $f(2), f(3), f(4)$ in terms of $f(1)$.

(2) Check the values of $f(1), f(2), f(3), f(4)$ whether it lies in given range or not.

15. Option (4) is correct.

Given: $y = y(t)$ is a solution of $\dfrac{dy}{dt} + \alpha y = \gamma e^{-\beta t}$

The given differential equation is a linear differential equation of the form $\dfrac{dy}{dx} + P(x)y = Q(x)$

Where I.F. $= e^{\int P(x)dx}$

and $y = \dfrac{1}{\text{I.F.}}\left[\int (\text{I.F.})Q(x)dx + C\right]$

So, here I.F. $= e^{\int \alpha\, dt}$.

$\Rightarrow$ I.F. $= e^{\alpha t}$

and $y = \dfrac{1}{e^{\alpha t}}\left[\int (e^{\alpha t})(\gamma e^{-\beta t})dt + C\right]$

$$\Rightarrow y = \frac{1}{e^{\alpha t}}\left[\gamma(e^{(\alpha-\beta)t}dt + C\right]$$

$$\Rightarrow ye^{\alpha t} = \gamma\frac{e^{(\alpha-\beta)t}}{\alpha-\beta} + C$$

$$\Rightarrow y = \frac{\gamma}{(\alpha-\beta)}\times\frac{e^{(\alpha-\beta)t}}{e^{\alpha t}} + \frac{C}{e^{\alpha t}}$$

$$\Rightarrow y = \frac{\gamma}{(\alpha-\beta)}e^{-\beta t} + Ce^{-\alpha t}$$

Now, $\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left[\dfrac{\gamma}{\alpha-\beta}e^{-\beta t} + Ce^{-\alpha t}\right]$

$$= \lim_{t\to\infty}\left[\frac{\gamma}{\alpha-\beta}\times\frac{1}{e^{\beta t}} + \frac{C}{e^{\alpha t}}\right] = 0$$

$\Rightarrow \lim_{t\to\infty} y(t) = 0$

16. Option (3) is correct.

Given : $\left|\dfrac{z-2i}{z+i}\right| = 2$

Let $z = p + iq$

$\Rightarrow \left|\dfrac{p+iq-2i}{p+iq+i}\right| = 2$

$\Rightarrow \left|\dfrac{p+i(q-2)}{p+i(q+1)}\right| = 2$

$\Rightarrow p^2 + (q-2)^2 = (2)^2[p^2 + (q+1)^2]$

$\Rightarrow p^2 + q^2 + 4 - 4q = 4p^2 + 4q^2 + 4 + 8q$

$\Rightarrow 3p^2 + 3q^2 + 12q = 0$

$\Rightarrow p^2 + q^2 + 4q = 0$

This is the equation of the circle of the form $x^2 + y^2 + 2gx + 2fy + c = 0$, where centre $\equiv (-g, -f)$ and radius $\equiv a^2 = g^2 + f^2 - C$

So, here $g \equiv 0, f \equiv 2$

$\therefore$ Centre $\equiv (0, -2)$

17. Option (1) is correct.

Given: A is symmetric matrix and B, C are skew symmetric matrices

$\Rightarrow A^T = A, B^T = -B, C^T = -C$

Statement 1: $A^{13}B^{26} - B^{26}A^{13}$ is symmetric

Now, $(A^{13}B^{26} - B^{26}A^{13})^T = (A^{13}B^{26})^T - (B^{26}A^{13})^T$

$= (B^{26})^T(A^{13})^T - (A^{13})^T(B^{26})^T \qquad \{\because (MN)^T = N^TM^T\}$

$= (B^T)^{26}(A^T)^{13} - (A^T)^{13}(B^T)^{26}$

$= (-B^T)^{26}(A)^{13} - (A)^{13}(-B)^{26}$

$= B^{26}A^{13} - A^{13}B^{26}$

$= -(A^{13}B^{26} - B^{26}A^{13})$

$\therefore A^{13}B^{26} - B^{26}A^{13}$ is not a symmetric matrix.

Statement 2: $A^{26}C^{13} - C^{13}A^{26}$ is symmetric

Now, $(A^{26}C^{13} - C^{13}A^{26})^T = (A^{26}C^{13})^T - (C^{13}A^{26})^T$

$= (C^{13})^T(A^{26})^T - (A^{26})^T(C^{13})^T$

$= (C^T)^{13}(A^T)^{26} - (A^T)^{26}(C^T)^{13}$

$= (-C^T)^{13}(A)^{26} - (A)^{26}(-C)^{13}$

$= -C^{13}A^{26} + A^{26}C^{13}$

$= A^{26}C^{13} - C^{13}A^{26}$

$\therefore A^{26}B^{13} - C^{13}A^{26}$ is a symmetric matrix.

18. Option (3) is correct.

We have to use digits 1, 3, 5, 7, 9 without repetition. The numbers should be greater than 5000 and less than 10000.

$\Rightarrow 5001 \le$ Number ≤ 9999

Let the number be $wxyz$

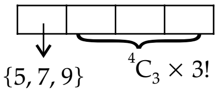
w can be 5, 7, 9 $\Rightarrow$ 3 options

x, y, z can be 4 digits $\Rightarrow {}^4C_3 \times 3!$

Total number of numbers $= 3 \times {}^4C_3 \times 3!$

$= 3 \times 4 \times 3 \times 2 = 72$

HINT:



be 4 digits number.

19. Option (1) is correct.

$f(x) = \log_{\sqrt{m}}(\sqrt{2}(\sin x - \cos x) + m - 2)$

As we know, the range of $a \sin x + b \cos x$ is $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$

$\Rightarrow$ Range of $\sin x - \cos x = [-\sqrt{2}, \sqrt{2}]$

$\Rightarrow -2 \le \sqrt{2}(\sin x - \cos x) \le 2$

$\Rightarrow -2 + m - 2 \le \sqrt{2}(\sin x - \cos x) + m - 2 \le 2 + m - 2$

$\Rightarrow m - 4 \le \sqrt{2}(\sin x - \cos x) + m - 2 \le m$

$\Rightarrow \log_{\sqrt{m}}(m-4) \le \log_{\sqrt{m}}[\sqrt{2}(\sin x - \cos x) + m - 2] \le \log_{\sqrt{m}} m$

$\Rightarrow \log_{\sqrt{m}}(m-4) = 0$

$\Rightarrow m - 4 = 1 \qquad \{\because \log_a 1 = 0\}$

$\Rightarrow m = 5$

HINT:

The range of $a\sin x + b\cos x$ is $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$

20. Option (2) is correct.

Given lines $L_1 : x + 1 = 2y = -12z$

$\Rightarrow \dfrac{x+1}{1} = \dfrac{y}{\frac{1}{2}} = \dfrac{z}{\frac{-1}{12}}$

and $L_2 : x = y + 2 = 6z - 6$

$\Rightarrow \dfrac{x}{1} = \dfrac{y+2}{1} = \dfrac{z-1}{\frac{1}{6}}$

Lines L_1 can be written as $\vec{r} = (-\hat{i}) + \lambda\left(\hat{i} + \frac{1}{2}\hat{j} - \frac{1}{12}\hat{k}\right)$

Lines L_2 can be written as $\vec{r} = (-2\hat{j} + \hat{k}) + \mu\left(\hat{i} + \hat{j} + \frac{1}{6}\hat{k}\right)$

As we know shortest distance between two lines $\vec{r} = \vec{a} + \lambda\vec{p}$ and $\vec{r} = \vec{b} + \mu\vec{q}$ is given by

$d = \left|\dfrac{(\vec{b}-\vec{a}).(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$

So, $\vec{a} = -\hat{i}, \vec{b} = -2\hat{j} + \hat{k}$,

$\vec{p} = \hat{i} + \frac{1}{2}\hat{j} - \frac{1}{12}\hat{k}, \vec{q} = \hat{i} + \hat{j} + \frac{1}{6}\hat{k}$

Now, $\vec{b} - \vec{a} = \hat{i} - 2\hat{j} + \hat{k}$

$$\vec{p}\times\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \frac{1}{2} & -\frac{1}{12} \\ 1 & 1 & \frac{1}{6} \end{vmatrix} = \frac{1}{6}\hat{i} - \frac{1}{4}\hat{j} + \frac{1}{2}\hat{k}$$

$\Rightarrow |\vec{p}\times\vec{q}| = \sqrt{\frac{1}{36}+\frac{1}{16}+\frac{1}{4}} = \frac{7}{12}$

Now, $(\vec{b}-\vec{a}).(\vec{p}\times\vec{q}) = \frac{1}{6}+\frac{1}{2}+\frac{1}{2} = \frac{7}{6}$

So, shortest distance $d = \left|\frac{\frac{7}{6}}{\frac{7}{12}}\right| = 2$

Section B

21. Correct answer is [9].

Let the event of a person being a smoker = S

and the event of a person having a lung cancer = C

$\Rightarrow P(S) = \frac{25}{100} = \frac{1}{4}$

$\Rightarrow P(\bar{S}) = 1-\frac{1}{4} = \frac{3}{4}$

Now, given that $P\left(\frac{C}{S}\right) = 27P\left(\frac{C}{\bar{S}}\right)$

Using Baye's theorem and law of total probability,

$$P\left(\frac{S}{C}\right) = \frac{P(S).P\left(\frac{C}{S}\right)}{P(S).P\left(\frac{C}{S}\right)+P(\bar{S}).P\left(\frac{C}{\bar{S}}\right)}$$

$$\Rightarrow \frac{k}{10} = \frac{\frac{1}{4}P\left(\frac{C}{S}\right)}{\frac{1}{4}P\left(\frac{C}{S}\right)+\frac{3}{4}.\frac{1}{27}P\left(\frac{C}{S}\right)}$$

$$\Rightarrow \frac{k}{10} = \frac{1}{1+\frac{1}{9}}$$

$$\Rightarrow \frac{k}{10} = \frac{9}{10} \Rightarrow k = 9$$

22. Correct answer is [7].

$(2023)^{22023} = (2030-7)^{2023}$

$= (35\lambda-7)^{2023}$

$= {}^{2023}C_0(35\lambda)^{2023} - {}^{2023}C_1(35\lambda)^{2022}(7) + ... - {}^{2023}C_{2023}7^{2023}$

$= 35\alpha - 7^{2023}$

Now, $-7^{2023} = -7(7)^{2022}$

$= -7(7^2)^{1011}$

$= -7(50-1)^{1011}$

$= -7[{}^{1011}C_0 50^{1011} - {}^{1011}C_1 50^{1010} + ... - {}^{1011}C_{1011}]$

$= -7[50\gamma-1] = -7[5\mu-1] = -35\mu + 7$

So, $(2023)^{2023} = 35\alpha - 35\mu + 7$

$\therefore$ When $(2023)^{2023}$ is divided by 35, the remainder is 7

23. Correct answer is [45].

Given: $a \in R$, α, β are roots of $x^2 + 60^{1/4}x + a = 0$

$\Rightarrow$ Sum of roots $= \alpha+\beta = -60^{1/4}$

and product of roots $= \alpha\beta = a$

[If α, β are roots of $ax^2+bx+c=0$ then, $\alpha+\beta = -\frac{b}{a}$ and $\frac{c}{a}$}

Now, $(\alpha+\beta)^2 = (-60^{1/4})^2$

$\Rightarrow \alpha^2+\beta^2+2\alpha\beta = 60^{1/2}$

$\Rightarrow \alpha^2+\beta^2 = 60^{1/2}-2a$

Squaring both sides of the above equation, we get

$(\alpha^2+\beta^2)^2 = (60^{1/2}-2a)^2$

$\Rightarrow \alpha^4+\beta^4+2\alpha^2\beta^2 = 60+4a^2-4(60)^{1/2}a$

$\Rightarrow \alpha^4+\beta^4 = 60+4a^2-4a(60)^{1/2}-2a^2$

$\Rightarrow \alpha^4+\beta^4 = 60+2a^2-4a(60)^{1/2}$

$\Rightarrow -30 = 60+2a^2-4a(60)^{1/2}$

$\Rightarrow 2a^2-4a(60)^{1/2}+90 = 0$

$\Rightarrow a^2-2a(60)^{1/2}+45 = 0$

The above equation is also a quadratic equation of the form $ax^2+bx+c=0$ whose product of roots $= \frac{c}{a}$

$\Rightarrow$ Product of all possible values of $a = 45$

24. Correct answer is [3].

Given: $a, b, \frac{1}{18}$ are in G.P.

and $\frac{1}{a}, 10, \frac{1}{b}$ are in A.P.

$\Rightarrow b^2 = \frac{a}{18}$ {$\because$ If a, b, c are in G.P. then $b^2 = ac$}

Similarly,

$\frac{1}{a}+\frac{1}{b} = 2(10)$ {$\because$ If a, b, c are in A.P. then $2b = a+c$}

$\Rightarrow \frac{1}{a} = 20-\frac{1}{b}$

$\Rightarrow \frac{1}{a} = \frac{20b-1}{b} \Rightarrow a = \frac{b}{20b-1}$

$\Rightarrow b^2 = \left(\frac{b}{20b-1}\right)\frac{1}{18}$

$\Rightarrow 18b^2(20b-1) = b$

$\Rightarrow 360b^2-18b-1 = 0$

$\Rightarrow 360b^2-30b+12b-1 = 0$

$\Rightarrow 30b(12b-1)+1(12b-1) = 0$

$\Rightarrow (30b+1)(12b-1) = 0$

$\Rightarrow b = \frac{1}{12}, \frac{-1}{30}$ (rejected) {$\because$ b is positive number}

$\Rightarrow a = \frac{1}{8}$

$\Rightarrow 16a+12b = 16\left(\frac{1}{8}\right)+12\left(\frac{1}{12}\right)$

$= 2+1 = 3$

25. Correct answer is [25].

Given $\cos 2\theta \cos\frac{\theta}{2} = \cos 3\theta.\cos\frac{9\theta}{2}$

$\Rightarrow 2\cos 2\theta.\cos\frac{\theta}{2} = 2\cos 3\theta.\cos\frac{9\theta}{2}$

As we know 2 cosA cos B = cos (A + B) + cos (A – B)

$\Rightarrow \cos\frac{5\theta}{2} + \cos\frac{3\theta}{2} = \cos\frac{15\theta}{2} + \cos\frac{3\theta}{2}$

$\Rightarrow \cos\frac{5\theta}{2} - \cos\frac{15\theta}{2} = 0$

As we know $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right).\sin\left(\frac{D-C}{2}\right)$

$\Rightarrow 2\sin 5\theta.\sin\frac{5\theta}{2} = 0$

$\Rightarrow \sin 5\theta = 0$ or $\sin\frac{5\theta}{2} = 0$

$\Rightarrow \theta = \frac{n\pi}{5}$ or $\frac{2n\pi}{5}$

$\Rightarrow \theta = 0, \pm\frac{\pi}{5}, \pm\frac{2\pi}{5}, \pm\frac{3\pi}{5}, \frac{4\pi}{5}, \pm\pi$

$\therefore m = 5$ and $n = 5$

So, $mn = 25$

26. Correct answer is [18].

Given : Line L_1: $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-2}{0}$

Equation of line can be written as

$\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \lambda(2\hat{i} - \hat{j})$

Equation of line passing through (1, 2, 3) and (2, 3, 4) is given by $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \mu(\hat{i} + \hat{j} + \hat{k})$

As we know shortest distance between two lines $\vec{r} = \vec{a} + \lambda\vec{p}$ and $\vec{r} = \vec{b} + \mu\vec{q}$ is given by

$$d = \left|\frac{(\vec{b} - \vec{a}).(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$$

So, $\vec{b} - \vec{a} = 3\hat{j} + \hat{k}$

$$\vec{p}\times\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$= -\hat{i} - 2\hat{j} + 3\hat{k}$

$\Rightarrow |\vec{p}\times\vec{q}| = \sqrt{1+4+9} = \sqrt{14}$

So, $(\vec{b} - \vec{a}).(\vec{p}\times\vec{q}) = 0 - 6 + 3 = -3$

So, shortest distance $a = \left|\frac{-3}{\sqrt{14}}\right| = \frac{3}{\sqrt{14}}$

$\therefore 28a^2 = 28 \times \frac{9}{14} = 18$

27. Correct answer is [3].

Given points P(–3, 2), Q(9, 10), R(α, 4) where PR is a diameter of the circle.

Using diameter form of circle, we can write the equation of circle $(x + 3)(x - \alpha) + (y - 2)(y - 4) = 0$

Now, Q (9, 10) lies on the above circle.

$\Rightarrow (9 + 3)(9 - \alpha) + (10 - 2)(10 - 4) = 0$

$\Rightarrow 12(9 - \alpha) + (8)(6) = 0$

$\Rightarrow 9 - \alpha = -\frac{48}{12} = -4$

$\Rightarrow \alpha = 13$

$\therefore$ Equation of circle is $(x + 3)(x - 13) + (y - 2)(y - 4) = 0$

$\Rightarrow x^2 + 3x - 13x - 39 + y^2 - 2y - 4y + 8 = 0$

$\Rightarrow x^2 + y^2 - 10x - 6y - 31 = 0$

Now, equation of tangent at Q (9, 10) is

$x(9) + y(10) - 5(x + 9) - 3(y + 10) - 31 = 0$

$\Rightarrow 9x + 10y - 5x - 45 - 3y - 30 - 31 = 0$

$\Rightarrow 4x + 7y = 106$ ...(1)

Similarly, equation of tangent at R (13, 4) is

$x(13) + y(4) - 5(x + 13) - 3(y + 4) - 31 = 0$

$\Rightarrow 13x + 4y - 5x - 65 - 3y - 12 - 31 = 0$

$\Rightarrow 8x + y = 108$ ...(2)

Now, S is the intersection point of both tangents

$\therefore$ From equation (1) and (2), we get

$8x + 14y = 212$ and $8x + y = 108$

$\Rightarrow 13y = 104$

$\Rightarrow y = 8$ and $x = \frac{25}{2}$

So point S is $\left(\frac{25}{2}, 8\right)$

Now, S lies on $2x - ky = 1$

$\Rightarrow 2\left(\frac{25}{2}\right) - 8k = 1$

$\Rightarrow 8k = 24 \Rightarrow k = 3$

28. Correct answer is [6860].

Given: 7 red, 5 white apples, 8 oranges

Let red apples be denoted by R

white apples be denoted by W

and oranges be denoted by O.

Then for selecting 5 fruits such that at least 2O, at least 1R, at least 1W be selected, the possible cases are

(1) 2O, 1R, 2W

(2) 2O, 2R, 1W

(3) 3O, 1R, 1W

Total number of ways = ${}^8C_2 . {}^7C_1 . {}^5C_2 + {}^8C_2 . {}^7C_2 . {}^5C_1 + {}^8C_3 . {}^7C_1 . {}^5C_1$

$$= \left(\frac{8\times7}{2}\times7\times\frac{5\times4}{2}\right) + \left(\frac{8\times7}{2}\times\frac{7\times6}{2}\times5\right) + \left(\frac{8\times7\times6}{3\times2}\times7\times5\right)$$

= 1960 + 2940 + 1960 = 6860

HINT:

Three possible cases are : 2O, 1R, 2W + 2O, 2R, 1W + 3O, 1R, 1W.

29. Correct answer is (20).

Let $I = \int_{\frac{1}{3}}^{3} |\log_e x| dx$

$$\because |\log_e x| = \begin{cases} -\log_e x, & \frac{1}{3} < x < 1 \\ \log_e x, & 1 \le x < 3 \end{cases}$$

$$\Rightarrow I = \int_{\frac{1}{3}}^{1} -\log_e x\, dx + \int_{1}^{3} \log_e x\, dx$$

$$\Rightarrow I = -\int_{\frac{1}{3}}^{1} \log_e x\, dx + \int_{1}^{3} \log_e x\, dx$$

Let $I_1 = \int \log_e x\, dx$

$$\Rightarrow I_1 = \int (\log_e x.1) dx$$

Now, using integration by parts

$$I_1 = \log_e x \int 1.dx - \int \left(\frac{d}{dx} \log_e x \int 1.dx \right) dx$$

$$\Rightarrow I_1 = (\log_e x)(x) - \int \left(\frac{1}{x} x \right) dx$$

$\Rightarrow I_1 = x \log_e x - x + c$

$$\therefore I = -[x \log_e x - x]_{\frac{1}{3}}^{1} + [x \log_e x - x]_1^3$$

$$\Rightarrow I = -\left(\log_e 1 - 1 - \frac{1}{3} \log_e \left(\frac{1}{3} \right) + \frac{1}{3} \right) + (3 \log_e 3 - 3 - \log_e 1 + 1)$$

$$\Rightarrow I = \frac{1}{3} \log_e \left(\frac{1}{3} \right) + \frac{2}{3} + 3 \log_e 3 - 2$$

$$\Rightarrow I = -\frac{1}{3} \log_e 3 + 3 \log_e 3 - 2 \log_e e + \frac{2}{3} \log_e e \quad \{\because \log_e e = 1\}$$

$$\Rightarrow I = \frac{8}{3} \log_e 3 - \frac{4}{3} \log_e e$$

$$\Rightarrow I = \frac{4}{3} (\log_e (3)^2 - \log_e e) \qquad \{\because a \log_b c = \log_b c^a\}$$

$$\Rightarrow I = \frac{4}{3} \log_e \left(\frac{9}{e} \right) \qquad \left\{\because \log_a b - \log_a c = \log_a \left(\frac{b}{c} \right)\right\}$$

$$\Rightarrow \frac{m}{n} \log_e \left(\frac{n^2}{e} \right) = \frac{4}{3} \log_e \left(\frac{9}{e} \right)$$

$\because$ m and n are co prime natural numbers

$\therefore$ $m = 4$ and $n = 3$

$\Rightarrow m^2 + n^2 - 5 = 4^2 + 3^2 - 5$

$= 16 + 9 - 5 = 20$

HINT:

(1) Use $|\log_e x| = \begin{cases} -\log_e x, & \frac{1}{3} < x < 1 \\ \log_e x, & 1 \le x < 3 \end{cases}$

(2) Use $\int \log_e x = x \log_e x - x + c$

(3) Use properties of logarithm for simplification.

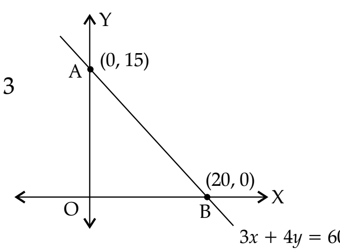
30. Correct answer is [31].

Given: A triangle is formed by x axis, y axis and $3x + 4y = 60$.

Coordinates of A are (0, 15) and B are (20, 0)

Now, P (a, b) is such that $a \in z$, $b = ka$

where $k \in z$



For $x = 1$, $4y = 60 - 3$

$$\Rightarrow y = \frac{57}{4}$$

$\Rightarrow y = 14.2$

For $a \in z$, $b = ka$, $k \in z$, $y = 1, 2, 3, 4, ..., 13, 14$

So 14 points lies in the triangle

For $x = 2$, $4y = 60 - 3$ (2)

$$\Rightarrow y = \frac{54}{4} \Rightarrow y = 13.5$$

For $x = 2$, $y = 2, 4, 6, 8, 10, 12$ so 6 points possible

Similarly, for $x = 3$, $y = 12.75$.

$\therefore$ For $x = 3$, $y = 3, 6, 9, 2$ so 4 points possible

For $x = 4$, $y = 4, 8$ so 2 points possible

For $x = 5$, $y = 5, 10$ so 2 points possible

For $x = 6$, $y = 6$ so only 1 point possible

For $x = 7$, $y = 7$ so only 1 point possible

For $x = 8$, $y = 8$ so only 1 point possible

Total number of points possible

$= 6 + 4 + 2 + 2 + 1 + 1 + 1 + 14 = 31$ points

HINT:

Find the vertices of the triangle, then for integer values of x, find y and check the conditions.

❑❑