JEE (Main) MATHEMATICS SOLVED PAPER

2023
29^{th} Jan. Shift 1

Section A

Q. 1. Let α and β be real numbers. Consider a 3 × 3 matrix A such that $A^2 = 3A + \alpha I$. If $A^4 = 21A + \beta I$, then

(1) $\beta = -8$ **(2)** $\beta = 8$ **(3)** $\alpha = 4$ **(4)** $\alpha = 1$

Q. 2. Let $x = 2$ be a root of the equation $x^2 + px + q = 0$ and

$$f(x) = \begin{cases} \dfrac{1-\cos(x^2-4px+q^2+8q+16)}{(x-2p)^4} & , x \neq 2p \\ 0 & , x = 2p \end{cases}$$

$\lim_{x\to 2p^+} [f(x)]$, where [.] denotes greatest integer function, is

(1) 0 **(2)** –1 **(3)** 2 **(4)** 1

Q. 3. Let B and C be the two points on the line $y + x = 0$ such that B and C are symmetric with respect to the origin. Suppose A is a point on $y - 2x = 2$ such that ΔABC is an equilateral triangle. Then, the area of the ΔABC is

(1) $\frac{10}{\sqrt{3}}$ **(2)** $3\sqrt{3}$ **(3)** $2\sqrt{3}$ **(4)** $\frac{8}{\sqrt{3}}$

Q. 4. Consider the following system of equations

$\alpha x + 2y + z = 1$
$2\alpha x + 3y + z = 1$
$3x + \alpha y + 2z = \beta$

for some $\alpha, \beta \in \mathbb{R}$. Then which of the following is NOT correct?

(1) It has a solution if $\alpha = -1$ and $\beta \neq 2$
(2) It has a solution for all $\alpha \neq -1$ and $\beta = 2$
(3) It has no solution for $\alpha = 3$ and for all $\beta \neq 2$
(4) It has no solution for $\alpha = -1$ and for all $\beta \in \mathbb{R}$

Q. 5. Let $y = f(x)$ be the solution of the differential equation $y(x + 1)\, dx - x^2 dy = 0$, $y(1) = e$. Then $\lim_{x\to 0^+} f(x)$ is equal to

(1) $\frac{1}{e^2}$ **(2)** e^2 **(3)** 0 **(4)** $\frac{1}{e}$

Q. 6. The domain of $f(x) = \dfrac{\log_{(x+1)}(x-2)}{e^{2\log_e x} - (2x+3)}$, $x \in \mathbb{R}$ is

(1) $\mathbb{R} - \{3\}$ **(2)** $(-1, \infty) - \{3\}$
(3) $(2, \infty) - \{3\}$ **(4)** $\mathbb{R} - \{-1, 3\}$

Q. 7. Fifteen football players of a club-team are given 15 T-shirts with their names written on the backside. If the players pick up the T-shirts randomly, then the probability that at least 3 players pick the correct T-shirt is

(1) $\frac{5}{24}$ **(2)** $\frac{1}{6}$ **(3)** $\frac{5}{36}$ **(4)** $\frac{2}{25}$

Q. 8. Let $[x]$ denote the greatest integer $\leq x$. Consider the function $f(x) = \max\{x^2, 1 + [x]\}$. Then the value of the integral $\int_0^2 f(x)dx$ is

(1) $\frac{5+4\sqrt{2}}{3}$ **(2)** $\frac{4+5\sqrt{2}}{3}$
(3) $\frac{1+5\sqrt{2}}{3}$ **(4)** $\frac{8+4\sqrt{2}}{3}$

Q. 9. For two non-zero complex numbers z_1 and z_2, if $\text{Re}(z_1z_2) = 0$ and $\text{Re}(z_1 + z_2) = 0$, then which of the following are possible?

A. $\text{Im}(z_1) > 0$ and $\text{Im}(z_2) > 0$
B. $\text{Im}(z_1) < 0$ and $\text{Im}(z_2) > 0$
C. $\text{Im}(z_1) > 0$ and $\text{Im}(z_2) < 0$
D. $\text{Im}(z_1) < 0$ and $\text{Im}(z_2) < 0$

Choose the correct answer from the options given below:

(1) B and D **(2)** A and B
(3) B and C **(4)** A and C

Q. 10. If the vectors $\vec{a} = \lambda\hat{i} + \mu\hat{j} + 4\hat{k}, b = -2\hat{i} + 4\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} + 3\hat{j} + \hat{k}$ are coplanar and the projection of $\vec{a}$ on the vector $\vec{b}$ is $\sqrt{54}$ units, then the sum of all possible values of $\lambda + \mu$ is equal to

(1) 0 **(2)** 24 **(3)** 6 **(4)** 18

Q. 11. Let $f(\theta) = 3\left(\sin^4\left(\frac{3\pi}{2} - \theta\right) + \sin^4(3\pi + \theta)\right) - 2(1 - \sin^2 2\theta)$ and $S = \left\{\theta \in [0, \pi]: f'(\theta) = -\frac{\sqrt{3}}{2}\right\}$. If $4\beta = \sum_{\theta\in S} \theta$, then $f(\beta)$ is equal to

(1) $\frac{5}{4}$ **(2)** $\frac{3}{2}$ **(3)** $\frac{9}{8}$ **(4)** $\frac{11}{8}$

Q. 12. If p, q and r be three propositions, then which of the following combination of truth values of p, q and r makes the logical expression $\{(p \vee q) \wedge ((\sim p) \vee r)\} \rightarrow ((\sim q) \vee r)$ false?

(1) p = T, q = T, r = F **(2)** p = T, q = F, r = T
(3) p = F, q = T, r = F **(4)** p = T, q = F, r = F

Q. 13. Let Δ be the area of the region $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 21, y^2 \leq 4x, x \geq 1\}$.

Then $\frac{1}{2}\left(\Delta - 21\sin^{-1}\frac{2}{\sqrt{7}}\right)$ is equal to

(1) $2\sqrt{3} - \frac{2}{3}$ **(2)** $\sqrt{3} - \frac{4}{3}$
(3) $\sqrt{3} - \frac{2}{3}$ **(4)** $2\sqrt{3} - \frac{1}{3}$

Q. 14. A light ray emits from the origin making an angle 30° with the positive x-axis. After getting reflected by the line $x + y = 1$, if this ray intersects x-axis at Q, then the abscissa of Q is

(1) $\frac{\sqrt{3}}{2(\sqrt{3}+1)}$ (2) $\frac{2}{3+\sqrt{3}}$

(3) $\frac{2}{(\sqrt{3}-1)}$ (4) $\frac{2}{3-\sqrt{3}}$

Q. 15. Let A = $\{(x, y) \in \mathbb{R}^2 : y \geq 0, 2x \leq y \leq$ and B = $\{(x,y) \in \mathbb{R} \times \mathbb{R} : 0 \leq y \leq \min\left\{2x, \sqrt{4-(x-1)^2}\right\}$.

Then the ratio of the area of A to the area of B is

(1) $\frac{\pi+1}{\pi-1}$ (2) $\frac{\pi}{\pi-1}$ (3) $\frac{\pi-1}{\pi+1}$ (4) $\frac{\pi}{\pi+1}$

Q. 16. Let $\lambda \neq 0$ be a real number. Let α, β be the roots of the equation $14x^2 - 31x + 3\lambda = 0$ and α, γ be the roots of the equation $35x^2 - 53x + 4\lambda = 0$. Then $\frac{3\alpha}{\beta}$ and $\frac{4\alpha}{\gamma}$ are the roots of the equation

(1) $49x^2 - 245x + 250 = 0$
(2) $7x^2 + 245x - 250 = 0$
(3) $7x^2 - 245x + 250 = 0$
(4) $49x^2 + 245x + 250 = 0$

Q. 17. Let the tangents at the points A(4, −11) and B(8, −5) on the circle $x^2 + y^2 - 3x + 10y - 15 = 0$, intersect at the point C. Then the radius of the circle, whose centre is C and the line joining A and B is its tangent, is equal to

(1) $2\sqrt{13}$ (2) $\sqrt{13}$ (3) $\frac{3\sqrt{3}}{4}$ (4) $\frac{2\sqrt{13}}{3}$

Q. 18. Let $f(x) = x + \frac{a}{\pi^2-4}\sin x + \frac{b}{\pi^2-4}\cos x, x \in \mathbb{R}$ be a function which satisfies $f(x) = x + \int_0^{\pi/2} \sin(x + y) f(y)dy$. Then $(a + b)$ is equal to

(1) $-2\pi(\pi - 2)$ (2) $-2\pi(\pi + 2)$
(3) $-\pi(\pi - 2)$ (4) $-\pi(\pi + 2)$

Q. 19. Let $f : R \to R$ be a function such that $f(x) = \frac{x^2+2x+1}{x^2+1}$. Then

(1) $f(x)$ is one-one in $[1, \infty)$ but not in $(-\infty, \infty)$
(2) $f(x)$ is one-one in $(-\infty, \infty)$
(3) $f(x)$ is many-one in $(-\infty, -1)$
(4) $f(x)$ is many-one in $(1, \infty)$

Q. 20. Three rotten apples are mixed accidently with seven good apples and four apples are drawn one by one without replacement. Let the random variable X denote the number of rotten apples. If μ and σ^2 represent mean and variance of X, respectively, then $10(\mu^2 + \sigma^2)$ is equal to

(1) 250 (2) 25 (3) 30 (4) 20

Section B

Q. 21. Let the co-ordinates of one vertex of Δ ABC be A(0, 2, α) and the other two vertices lie on the line $\frac{x+\alpha}{5} = \frac{y-1}{2} = \frac{z+4}{3}$. For $\alpha \in Z$, if the area of ΔABC is 21 sq. units and the line segment BC has length $2\sqrt{21}$ units, then α^2 is equal to

Q. 22. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function that satisfies the relation $f(x + y) = f(x) + f(y) - 1$, $\forall x, y \in \mathbb{R}$. If $f'(0) = 2$, then $|f(-2)|$ is equal to

Q. 23. Suppose f is a function satisfying $f(x + y) = f(x) + f(y)$ for all $x, y \in$ N and $f(1) = \frac{1}{5}$. If $\sum_{n=1}^{m} \frac{f(n)}{n(n+1)(n+2)} = \frac{1}{12}$, them m is equal to

Q. 24. Let the coefficients of three consecutive terms in the binomial expansion of $(1 + 2x)^n$ be in the ratio 2: 5: 8. Then the coefficient of the term, which is in the middle of these three terms, is

Q. 25. Let $a_1, a_2, a_3, \ldots$ be a GP of increasing positive numbers. If the product of fourth and sixth terms is 9 and the sum of fifth and seventh terms is 24 , then $a_1a_9 + a_2a_4a_9 + a_5 + a_7$ is equal to

Q. 26. Let the equation of the plane P containing the line $x + 10 = \frac{8-y}{2} = z$ be $ax + by + 3z = 2(a + b)$ and the distance of the plane P from the point (1, 27, 7) be c. Then $a^2 + b^2 + c^2$ is equal to

Q. 27. If the coefficient of x^9 in $\left(\alpha x^3 + \frac{1}{\beta x}\right)^{11}$ and the coefficient of x^{-9} in $\left(\alpha x - \frac{1}{\beta x^3}\right)^{11}$ are equal, then $(\alpha\beta)^2$ is equal to

Q. 28. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero non-coplanar vectors. Let the position vectors of four points A, B, C and D be $\vec{a} - \vec{b} + \vec{c}, \lambda\vec{a} - 3\vec{b} + 4\vec{c}, -\vec{a} + 2\vec{b} - 3\vec{c}$ and $2\vec{a} - 4\vec{b} + 6\vec{c}$ respectively. If $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ and coplanar, then λ is equal to

Q. 29. Five digit numbers are formed using the digits 1, 2, 3, 5, 7 with repetitions and are written in descending order with serial numbers. For example, the number 77777 has serial number 1. Then the serial number of 35337 is

Q. 30. If all the six digit numbers $x_1x_2x_3x_4x_5x_6$ with $0 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6$ are arranged in the increasing order, then the sum of the digits in the 72th number is

Answer Key

Q. No.	Answer	Topic Name	Chapter Name
1	(1)	Algebra of Matrices	Matrices and Determinants
2	(1)	Limit of Trigometric Functions	Limit, Continuity and Differentiability
3	(4)	Perpendicular bisector	Straight line and a point
4	(3)	Solution of linear equations	Matrices and Determinants
5	(3)	Variable seprable form	Differential Equations
6	(3)	Domain	Functions
7	(4)	Basics of probability	Probability
8	(1)	Properties of definite integration	Definite integration
9	(3)	Algebra of complex numbers	Complex numbers
10	(2)	Scalar Triple Product	Vector Algebra
11	(1)	Trigonometric equations	Trigonometry
12	(3)	Truth tables	Mathematical Reasoning
13	(2)	Area Bounded by Curves	Area under Curves
14	(2)	Interaction of two lines	Straight line and a point
15	(3)	Area Bounded by Curves	Area under Curves
16	(1)	Nature of roots	Quadratic Equations
17	(4)	Tangent and Normal	Circle
18	(2)	Properties of Definite integration	Definite Integration
19	(1)	Increasing and Decreasing	Application of derivative
20	(4)	Binomial distribution and probability distribution	Probability
21	[9]	Line	Three Dimensional Geometry
22	[3]	Integration	Integration
23	[10]	Functional Equation	Functions
24	[1120]	Binomial theorem for a positive integral index	Binomial Theorem
25	[60]	Geometric Progressions	Sequences and Series
26	[355]	Plane and a Point	Three Dimensional Geometry
27	[1]	Binomial theorem for a positive integral index	Binomial Theorem
28	[2]	Coplanarity of vectors	Vector Algebra
29	[1436]	Permutations	Permutations and Combinations
30	[32]	Combinations	Permutations and Combinations

Solutions

Section A

1. Option (1) is correct.

Given, $A^2 = 3A + \alpha I$ ...(i)

Consider, $A^4 = A^2A^2$

$\Rightarrow A^4 = (3A + \alpha I)\cdot(3A + \alpha I)$

$\Rightarrow A^4 = 9A^2 + 3\alpha IA + 3\alpha AI + \alpha^2 I^2$

$\Rightarrow A^4 = 9A^2 + 6\alpha A + \alpha^2 I^2$

$\Rightarrow A^4 = 9(3A + \alpha I) + 6\alpha A + \alpha^2 I$ using (i)

$\Rightarrow A^4 = (27 + 6\alpha)A + (9\alpha + \alpha^2)I$

Comparing with $A^4 = 21A + \beta I$, we get

$27 + 6\alpha = 21$

$\Rightarrow 6\alpha = -6 \Rightarrow \alpha = -1$

Also $\beta = 9\alpha + \alpha^2$

$\Rightarrow \beta = 9(-1) + (-1)^2 \Rightarrow \beta = -8$

HINT:

(1) Use $A^{m+n} = A^m \cdot A^n$

(2) $AI = IA = A$

(3) $I^n = I$

2. Option (1) is correct.

Since $x = 2$ is a root of $x^2 + px + q = 0$

$\Rightarrow 4 + 2p + q = 0$

$\Rightarrow 2p = -(q + 4)$ ...(i)

Now, $\lim_{x \to 2p^+} [f(x)]$

$$= \lim_{x\to 2p^+}\left[\frac{1-\cos(x^2-4px+q^2+8q+16)}{(x-2p)^4}\right]$$

$$= \lim_{x\to 2p^+}\left[\frac{1-\cos\left((x-2p)^2-4p^2+(q+4)^2\right)}{(x-2p)^4}\right]$$

$$= \lim_{x\to 2p^+}\left[\frac{1-\cos\left((x-2p)^2-4p^2+(-2p)^2\right)}{(x-2p)^4}\right] \text{ Using (i)}$$

$$= \lim_{x\to 2p^+}\left[\frac{1-\cos((x-2p)^2-4p^2+4p^2)}{(x-2p)^4}\right]$$

$$= \lim_{x\to 2p^+}\left[\frac{1-\cos(x-2p)^2}{((x-2p)^2)^2}\right]$$

$$= \left[\frac{1}{2}\right], \text{ as } \lim_{h\to 0}\frac{1-\cos h}{h^2}=\frac{1}{2}$$

So, $\lim_{x\to 2p^+}[f(x)]=0$

3. Option (4) is correct.

It is given that, B & C lies on $x + y = 0$

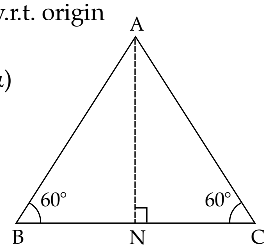
Also B & C are symmetric w.r.t. origin

$\Rightarrow N \equiv (0, 0)$

Let $B \equiv (-\alpha, \alpha) \Rightarrow C \equiv (\alpha, -\alpha)$

Slope of BC = –1

So, slope of AN = $\frac{-1}{(-1)}=1$



Now, equation of AN is,

$y - 0 = 1 (x - 0)$

$\Rightarrow y = x$

Since, A lies on $y - 2x = 2$ and also on $y = x$

$\Rightarrow x - 2x = 2$

$\Rightarrow x = -2, y = -2$

So, AN = $\sqrt{(0-(-2))^2+(0-(-2))^2}$

$\Rightarrow$ AN = $\sqrt{4+4}=2\sqrt{2}$

In ΔABN,

$\tan 60° = \frac{AN}{BN}$

$\Rightarrow BN = \frac{2\sqrt{2}}{\sqrt{3}}$

Area of $\Delta ABC = \frac{1}{2}(AN)(BC)=\frac{1}{2}(AN)(2BN)$

$= \frac{1}{2}(2\sqrt{2})\left(2\left(\frac{2\sqrt{2}}{\sqrt{3}}\right)\right) = \frac{8}{\sqrt{3}}$ sq. unit

HINT:

(1) Find equation of perpendicular bisector of BC and solve for point A.

(2) If two line having slope m_1 & m_2 are perpendicular, then $m_1m_2 = -1$

4. Option (3) is correct.

System of equation is,

$\alpha x + 2y + z = 1$

$2\alpha x + 3y + z = 1$

$3x + \alpha y + 2z = \beta$

Now, $D = \begin{vmatrix}\alpha & 2 & 1\\ 2\alpha & 3 & 1\\ 3 & \alpha & 2\end{vmatrix}$

$\Rightarrow D = \alpha^2 - 2\alpha - 3$

For no solution, D = 0

$\Rightarrow \alpha^2 - 2\alpha - 3 = 0$

$\Rightarrow (\alpha + 1)(\alpha - 3) = 0 \Rightarrow \alpha = -1, 3$

Here, $D_1 = \begin{vmatrix}1 & 2 & 1\\ 1 & 3 & 1\\ \beta & \alpha & 2\end{vmatrix}$

$\Rightarrow D_1 = 2 - \beta$

$D_2 = \begin{vmatrix}-1 & 1 & 1\\ -2 & 1 & 1\\ 3 & \beta & 2\end{vmatrix}$

$\Rightarrow D_2 = 2 - \beta$

$D_3 = \begin{vmatrix}-1 & 2 & 1\\ -2 & 3 & 1\\ 3 & -1 & \beta\end{vmatrix}$

$\Rightarrow D_3 = \beta - 2$

So, equation has no solution for,

$\beta \neq 2, \alpha = -1$ and $\beta \neq 2, \alpha = 3$

HINT:

System of equations will have no solution if D = 0 and if at least one of D_1, D_2 or D_3 is non-zero.

5. Option (3) is correct.

Given differential equation is,

$y(x + 1)\, dx - x^2 dy = 0$

$\Rightarrow y(x + 1)dx = x^2 dy$

$\Rightarrow \left(\frac{x+1}{x^2}\right)dx = \frac{dy}{y}$

$\Rightarrow \left(\frac{1}{x}+\frac{1}{x^2}\right)dx = \frac{dy}{y}$

Integrating both sides, we get

$\int\left(\frac{1}{x}+\frac{1}{x^2}\right)dx = \int\frac{1}{y}dy$

$\Rightarrow \ln x - \frac{1}{x} = \ln y + c$

Also, $y(1) = e$

$\therefore \ln(1) - \frac{1}{1} = \ln(e) + c$

$\Rightarrow c = -2$

So, $\ln(y) = \ln(x) - \frac{1}{x} + 2$

$\Rightarrow \ln\left(\frac{y}{x}\right) = 2 - \frac{1}{x}$

$$\Rightarrow \frac{y}{x} = e^{\left(2-\frac{1}{x}\right)} \quad \Rightarrow y = x\,e^{\left(2-\frac{1}{x}\right)}$$

Now, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} y$

$$= \lim_{x\to 0^+} x\,e^{\left(2-\frac{1}{x}\right)} = \lim_{x\to 0^+} e^2\left(\frac{x}{e^{\frac{1}{x}}}\right)$$

$= 0$, as $x \to 0^+, e^{\frac{1}{x}} \to \infty$

6. Option (3) is correct.

Given, $f(x) = \dfrac{\log_{(x+1)}(x-2)}{e^{2\log_e x} - (2x+3)}$

$x - 2 > 0 \Rightarrow x > 2$ ...(1)

$x + 1 > 0 \Rightarrow x > -1$ ...(2)

$x + 1 \neq 1 \Rightarrow x \neq 0$ ...(3)

$x > 0$ ...(4)

$$f(x) = \frac{\log_{(x+1)}(x-2)}{e^{2\log_e x} - (2x+3)}$$

$x^2 - 2x - 3 \neq 0$

$\Rightarrow (x + 1)(x - 3) \neq 0$

$\Rightarrow x \neq -1, 3$ ...(5)

From (1), (2), (3), (4) & (5)

$x \in (2, \infty) - \{3\}$

7. Option (4) is correct.

Probability (at least 3 players picked correct T-shirt)

$$= 1 - \begin{bmatrix} \text{Probability (All 15 picked wrong T-shirt)} \\ +\text{Probability (14 picked wrong T-shirt)} \\ +\text{Probability (13 picked wrong T-shirt)} \end{bmatrix}$$

$= 1 - (A + B + C)$

Let D_k represents number of ways when 'k' out of 'n' distinct objects/ envelopes are dearranged or wrongly placed.

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ...\right]$$

Also, $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + ...$

So, $e^{-x} = 1 - \dfrac{x}{1!} + \dfrac{x^2}{2!} - ....$

$$\Rightarrow D_n = \frac{n!}{e}$$

Now, $D_k = ({}^nC_k)\,\dfrac{k!}{e}$

A = probability (All 15 players wrongly picked)

$$= \left(\frac{15!}{e}\right)\cdot\frac{1}{15!}$$

B = Probability (14 players wrongly picked)

$$= \left({}^{15}C_{14}\left(\frac{14!}{e}\right)\right)\frac{1}{15!}$$

C = Probability (13 players wrongly picked)

$$= \left({}^{15}C_{13}\left(\frac{13!}{e}\right)\right)\frac{1}{15!}$$

So, probability (at least 3 players picked correct T-shirt)

$$= 1 - \frac{\frac{15!}{e} + ({}^{15}C_{14})\left(\frac{14!}{e}\right) + ({}^{15}C_{13})\left(\frac{13!}{e}\right)}{15!}$$

$$= 1 - \left(\frac{1}{e} + \frac{1}{e} + \frac{1}{2e}\right) = 1 - \frac{5}{2e} = 0.08 \approx \frac{2}{25}$$

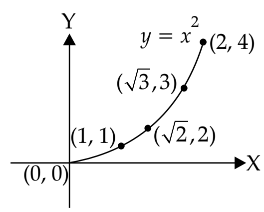
HINT:

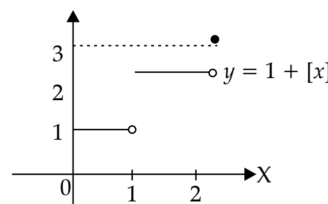
(1) $P(A^c) = 1 - P(A)$

(2) Dearrangement of 'k' envelopes out of 'n' envelopes $({}^nC_k)\left(\dfrac{k!}{e}\right)$.

8. Option (1) is correct.

Given, $f(x) = \max\{x^2, 1 + [x]\}$, $x \in [0, 2]$, where [.] ≡ GIF





So, $f(x) = \begin{cases} 1 + [x], & x \in [0, \sqrt{2}] \\ x^2 & x \in [\sqrt{2}, 2] \end{cases}$

Now, $\int_0^2 f(x)dx$

$$= \int_0^{\sqrt{2}} (1 + [x])dx + \int_{\sqrt{2}}^2 x^2 dx$$

$$= \int_0^1 (1+0)dx + \int_1^{\sqrt{2}} (1+1)dx + \left[\frac{x^3}{3}\right]_{\sqrt{2}}^2$$

$$\Rightarrow \int_0^2 f(x)dx = [x]_0^1 + [2x]_1^{\sqrt{2}} + \left(\frac{8}{3} - \frac{2\sqrt{2}}{3}\right)$$

$$\Rightarrow \int_0^2 f(x)dx = (1-0) + 2(\sqrt{2}-1) + \frac{1}{3}(8 - 2\sqrt{2})$$

$$= \frac{5 + 4\sqrt{2}}{3}$$

9. Option (3) is correct.

$\Rightarrow$ Let $z_1 = p_1 + iq_1$ & $z_2 = p_2 + iq_2$

$z_1 + z_2 = (p_1 + p_2) + i(q_1 + q_2)$

Since, Re $(z_1 + z_2) = 0$

$\Rightarrow p_1 + p_2 = 0$ ...(1)

Now, $z_1 z_2 = (p_1 + iq_1)(p_2 + iq_2)$

$\Rightarrow z_1 z_2 = p_1 p_2 + ip_1 q_2 + ip_2 q_1 + i^2 q_1 q_2$

$\Rightarrow z_1 z_2 = (p_1 p_2 - q_1 q_2) + i(p_1 q_2 + p_2 q_1)$

Since, Re $(z_1 z_2) = 0$

$\Rightarrow p_1 p_2 - q_1 q_2 = 0$

From (i) $p_2 = -p_1$
$\Rightarrow -p_1^2 - q_1q_2 = 0$
$\Rightarrow q_1q_2 = -p_1^2 < 0$, as $p_1 \in R$ & $q_1, q_2 \in R$
Since, $q_1q_2 < 0$
$\Rightarrow$ Im (z_1) & Im (z_2) are of opposite signs.

HINT:

(1) Use $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$

(2) If $x \in R_1$ then $x^2 \geq 0$

10. Option (2) is correct.

Given, $\vec{a} = \lambda\hat{i} + \mu\hat{j} + 4\hat{k}$

$\vec{b} = -2\hat{i} + 4\hat{j} - 2\hat{k}$

$\vec{c} = 2\hat{i} + 3\hat{j} + \hat{k}$

Since, $\vec{a}, \vec{b}$ & $\vec{c}$ are coplanar

$[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$$\Rightarrow \begin{vmatrix} \lambda & \mu & 4 \\ -2 & 4 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$\Rightarrow \lambda(4 + 6) - \mu(-2 + 4) + 4(-6 - 8) = 0$
$\Rightarrow 10\lambda - 2\mu - 56 = 0$
$\Rightarrow 5\lambda - \mu = 28$ ...(1)

Also, projection of $\vec{a}$ on $\vec{b}$ = $\sqrt{54}$ units

$$\Rightarrow \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \sqrt{54}$$

$$\Rightarrow \frac{(\lambda\hat{i} + \mu\hat{j} + 4\hat{k})\cdot(-2\hat{i} + 4\hat{j} + -2\hat{k})}{\sqrt{4 + 16 + 4}} = \sqrt{54}$$

$\Rightarrow -2\lambda + 4\mu - 8 = (3\sqrt{6})(2\sqrt{6})$

$\Rightarrow -2\lambda + 4\mu - 8 = 36$
$\Rightarrow -2\lambda + 4\mu = 44 \Rightarrow \lambda - 2\mu + 22 = 0$
From (1)

$\lambda - 2(5\lambda - 28) + 22 = 0 \Rightarrow \lambda = \frac{26}{3}$

From (1), $\mu = 5\lambda - 28 = \frac{130}{3} - 28$

$\Rightarrow \mu = \frac{46}{3}$

So $\lambda + \mu = \frac{26}{3} + \frac{46}{3} = \frac{72}{3} = 24$

HINT:

(1) If $\vec{p}, \vec{q}$ & $\vec{r}$ are coplanar, then $[\vec{p}\ \vec{q}\ \vec{r}] = 0$

(2) Projection of $\vec{\alpha}$ on $\vec{\beta}$ $= \frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\beta}|}$

11. Option (1) is correct.

Given $f(\theta) = 3\left(\sin^4\left(3\frac{\pi}{2} - \theta\right) + \sin^4(3\pi + \theta)\right) - 2(1 - \sin^2 2\theta)$

$\Rightarrow f(\theta) = 3(\cos^4\theta + (-\sin\theta)^4) - 2\cos^2 2\theta$
$\Rightarrow f(\theta) = 3(\cos^4\theta + \sin^4\theta) - 2\cos^2 2\theta$
$\Rightarrow f(\theta) = 3[(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta] - 2\cos^2 2\theta$

$$\Rightarrow f(\theta) = 3\left[1 - \frac{2}{4}(2\sin\theta\cos\theta)^2\right] - 2\cos^2 2\theta$$

$$\Rightarrow f(\theta) = 3 - \frac{3}{2}\sin^2 2\theta - 2\cos^2 2\theta$$

$$\Rightarrow f(\theta) = 3 - \frac{3}{2}(\sin^2 2\theta + \cos^2 2\theta) - \frac{\cos^2 2\theta}{2}$$

$$\Rightarrow f(\theta) = 3 - \frac{3}{2} - \frac{\cos^2 2\theta}{2}$$

$$\Rightarrow f(\theta) = \frac{3}{2} - \frac{\cos^2 2\theta}{2}$$

Now, $f'(\theta) = 0 - \frac{1}{2}(-(2\cos 2\theta)2\sin 2\theta)$

$\Rightarrow f'(\theta) = \sin 4\theta$

Given, $\sin 4\theta = -\frac{\sqrt{3}}{2}, \theta \in [0, \pi]$

$$\Rightarrow 4\theta = \pi + \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 3\pi + \frac{\pi}{3}, 4\pi - \frac{\pi}{3}$$

$$\Rightarrow 4\theta = \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}$$

$$\Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{12}, \frac{5\pi}{6}, \frac{11\pi}{12}$$

Now, $4\beta = \sum_{\theta \in S} \theta$

$$\Rightarrow 4\beta = \frac{\pi}{3} + \frac{5\pi}{12} + \frac{5\pi}{6} + \frac{11\pi}{12}$$

$$\Rightarrow 4\beta = \frac{5\pi}{2} \Rightarrow \beta = \frac{5\pi}{8}$$

So, $f(\beta) = f\left(\frac{5\pi}{8}\right) = \frac{3}{2} - \frac{1}{2}\cos^2\left(2\left(\frac{5\pi}{8}\right)\right)$

$$= \frac{3}{2} - \frac{1}{2}\cos^2\left(\frac{5\pi}{4}\right)$$

$$= \frac{3}{2} - \frac{1}{2}\left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{3}{2} - \frac{1}{4} = \frac{5}{4}$$

12. Option (3) is correct.

Let E = $((p \vee q) \wedge (\sim p) \vee r)) \rightarrow ((\sim q) \vee r)$
$E_1 = p \vee q$
$E_2 = \sim p \vee r$
$E_3 = \sim q \vee r$

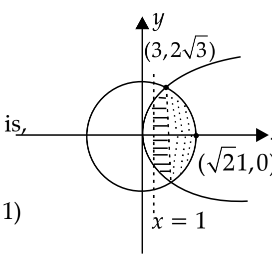
p	q	r	$\sim p$	$\sim q$	E_1	E_2	E_3
T	T	T	F	F	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	T	F	T
F	T	T	T	F	T	T	T
F	T	F	T	F	T	T	F
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

Now, $E = (E_1 \wedge E_2) \rightarrow E_3$
or $E = \sim (E_1 \wedge E_2) \vee E_3$
or $E = \sim E_1 \vee \sim E_2 \vee E_3$
$p = T, q = T, r = F$
$E_1 = T, E_2 = F, E_3 = F$
So, E = T
$p = T, q = F, r = T$
$E_1 = T, E_2 = T, E_3 = T$
So, E = T
$p = F, q = T, r = F$
$E_1 = T, E_2 = T, E_3 = F$
So, E = F

HINT:
(1) Draw truth table
(2) $p \rightarrow q = \sim p \vee q$

13. Option (2) is correct.

Converting the given inequations into equations, we get
$\Rightarrow x^2 + y^2 = 21, y^2 = 4x$
Solving both
$x^2 + 4x - 21 = 0$
$\Rightarrow (x-3)(x+7) = 0 \Rightarrow x = 3, -7$
But $y^2 = 4x \Rightarrow y^2 = -28$
So, $x = 3$



Area of shaded region

$$= 2\left[\int_1^3 2\sqrt{x}dx + \int_3^{\sqrt{21}} \sqrt{21 - x^2}dx\right]$$

$$= 4\left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^3 + 2\left\{\frac{21}{2}\left[\sin^{-1}\left(\frac{x}{\sqrt{21}}\right) + \frac{x}{2}\sqrt{21 - x^2}\right]_3^{\sqrt{21}}\right\}$$

$$= \frac{8}{3}(3\sqrt{3} - 1) + (21(\sin^{-1} 1) + 0) - \left(21\sin^{-1}\left(\frac{3}{\sqrt{21}}\right) + 3\sqrt{12}\right)$$

$$= 8\sqrt{3} - \frac{8}{3} + \frac{21\pi}{2} - 6\sqrt{3} - 21\sin^{-1}\left(\sqrt{\frac{3}{7}}\right)$$

$$\Rightarrow \text{Area} = 2\sqrt{3} + \frac{21\pi}{2} - \frac{8}{3} - 21\sin^{-1}\left(\sqrt{\frac{3}{7}}\right)$$

$$\text{Now, } \frac{1}{2}\left(\Delta - 21\sin^{-1}\left(\frac{2}{\sqrt{7}}\right)\right)$$

$$= \frac{1}{2}\left[2\sqrt{3} + 21\frac{\pi}{2} - \frac{8}{3} - 21\sin^{-1}\sqrt{\frac{3}{7}} - 21\sin^{-1}\left(\frac{2}{\sqrt{7}}\right)\right]$$

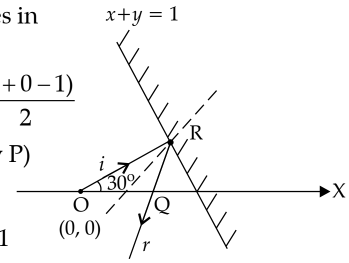
$$= \frac{1}{2}\left[2\sqrt{3} + \frac{21\pi}{2} - \frac{8}{3} - 21\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left(2\sqrt{3} - \frac{8}{3}\right) = \sqrt{3} - \frac{4}{3}$$

14. Option (2) is correct.

Incident Ray $(i) \equiv y = \frac{x}{\sqrt{3}}$

Image of O(0, 0) in line mirror $x + y = 1$, lies in reflected ray



$$\frac{x-0}{1} = \frac{y-0}{1} = -2\frac{(0+0-1)}{2}$$

$\Rightarrow (x, y) \equiv (1, 1)$ (say P)

for point R, solve

$y = \frac{x}{\sqrt{3}}$ & $x + y = 1$

$$\Rightarrow x + \frac{x}{\sqrt{3}} = 1$$

$$\Rightarrow \frac{\sqrt{3}+1}{\sqrt{3}}x = 1 \Rightarrow x = \frac{\sqrt{3}}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{3-\sqrt{3}}{3-1}$$

$$\Rightarrow x = \frac{3-\sqrt{3}}{2}$$

$$\Rightarrow y = \frac{3-\sqrt{3}}{2\sqrt{3}} = \frac{\sqrt{3}-1}{2}$$

$$\text{So, } R \equiv \left(\frac{3-\sqrt{3}}{2}, \frac{\sqrt{3}-1}{2}\right)$$

Equation of reflected Ray is,

$$y - 1 = \left(\frac{\left(\frac{\sqrt{3}-1}{2}\right) - 1}{\left(\frac{3-\sqrt{3}}{2}\right) - 1}\right)(x - 1)$$

$$\Rightarrow y - 1 = \left(\frac{\sqrt{3}-3}{1-\sqrt{3}}\right)(x-1)$$

$$\Rightarrow y - 1 = \sqrt{3}(x-1)$$

$$\Rightarrow y = \sqrt{3}x + (1 - \sqrt{3})$$

For point Q, put

$$y = 0 \Rightarrow \sqrt{3}x = \sqrt{3} - 1$$

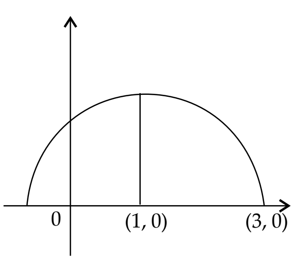
$$\Rightarrow x = 1 - \frac{1}{\sqrt{3}}$$

$$\text{or } x = \frac{\sqrt{3}-1}{\sqrt{3}}$$

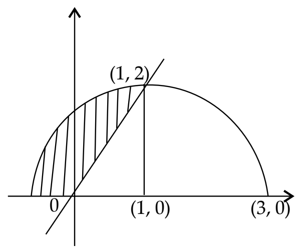
$$\text{or } x = \left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right) \text{ or } \alpha = \frac{2}{3+\sqrt{3}}$$

15. Option (3) is correct.

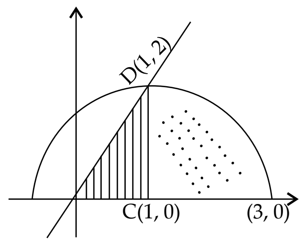
$C_1 : y \le \sqrt{4 - (x-1)^2}$
$\Rightarrow y^2 \le 4 - (x-1)^2$
$\Rightarrow (x-1)^2 + y^2 \le 4$
C_1 represents inner part of semi-circle (on or above x-axis) whose centre is (1, 0) & radius is 2 unit



Now, region A is



Now, region B is



Solving, $y = 2x$ & $(x-1)^2 + y^2 = 4$

$(x-1)^2 + (2x)^2 = 4$

$\Rightarrow x^2 - 2x + 1 + 4x^2 = 4$

$\Rightarrow 5x^2 - 2x - 3 = 0 \Rightarrow (5x+3)(x-1) = 0$

$\Rightarrow x = 1, -\dfrac{3}{5}$

Area of region B(shaded part)

= Area of ΔOCD + Area of (CDE)

$$= \left(\frac{1}{2}.1.2\right) + \int_1^3 \sqrt{4-(x-1)^2}\,dx$$

$$= 1 + \left[\left(\frac{x-1}{2}\right)\sqrt{4-(x-1)^2} + \frac{4}{2}\sin^{-1}\left(\frac{x-1}{2}\right)\right]_1^3$$

$$= 1 + \left[\begin{matrix}\left(\frac{3-1}{2}\right)\sqrt{4-(3-1)^2} + 2\sin^{-1}\left(\frac{3-1}{2}\right) \\ -0-0\end{matrix}\right]$$

$= 1 + 2\sin^{-1}(1)$

$$= 1 + 2\left(\frac{\pi}{2}\right) = (1+\pi) \text{ sq. units}$$

Now, area of region A (shaded part)

= Area of semi - circle – Area of (shaded part) B

$$= \frac{\pi(2)^2}{2} - (\pi+1) = (\pi - 1) \text{ sq. unit}$$

So, $\dfrac{A}{B} = \dfrac{\pi - 1}{\pi + 1}$

HINT:

(1) Draw rough sketch of given curves and shade the required part.

(2) Find point of intersection by solving both curves.

16. Option (1) is correct.

Roots of equation $14x^2 - 31x + 3\lambda = 0$ are α and β

Roots of equation $35x^2 - 53x + 4\lambda = 0$ are α and γ

Here α is a common root.

So, $\dfrac{\alpha^2}{-124\lambda + 159\lambda} = \dfrac{-\alpha}{56\lambda - 105\lambda} = \dfrac{1}{-742 + 1085}$

$\Rightarrow \dfrac{\alpha^2}{35\lambda} = \dfrac{\alpha}{49\lambda} = \dfrac{1}{343}$

Here, $\dfrac{\alpha}{49\lambda} = \dfrac{1}{343} \Rightarrow \alpha = \dfrac{\lambda}{7}$

Also, $\dfrac{\alpha^2}{35\lambda} = \dfrac{1}{343} \Rightarrow \alpha^2 = \dfrac{35\lambda}{343}$

So, $\dfrac{35\lambda}{343} = \left(\dfrac{\lambda}{7}\right)^2$

$\Rightarrow \dfrac{35\lambda}{343} = \dfrac{\lambda^2}{49} \Rightarrow \lambda^2 - 5\lambda = 0 \Rightarrow \lambda = 0, 5$

But $\lambda \neq 0 \Rightarrow \lambda = 5$

$\therefore\ \alpha = \dfrac{\lambda}{7} = \dfrac{5}{7}$

Also, $\alpha + \beta = \dfrac{31}{14}$

$\Rightarrow \beta = \dfrac{31}{14} - \dfrac{5}{7} = \dfrac{21}{14} = \dfrac{3}{2}$

Again, $\alpha + \gamma = \dfrac{53}{35}$

$\Rightarrow \gamma = \dfrac{53}{35} - \dfrac{5}{7} = \dfrac{28}{35} = \dfrac{4}{5}$

Now, $S = \dfrac{3\alpha}{\beta} + \dfrac{4\alpha}{\gamma} = \dfrac{10}{7} + \dfrac{25}{7} = \dfrac{35}{7}$

$P = \dfrac{3\alpha}{\beta}.\dfrac{4\alpha}{\gamma} = \dfrac{10}{7}.\dfrac{25}{7} = \dfrac{250}{49}$

Equation whose roots are

$\dfrac{3\alpha}{\beta}$ & $\dfrac{4\alpha}{\gamma}$ is

$x^2 - Sx + P = 0$

$\Rightarrow x^2 - \left(\dfrac{35}{7}\right)x + \dfrac{250}{49} = 0$

$\Rightarrow 49x^2 - 245x + 250 = 0$

17. Option (4) is correct.

Point A $\equiv (4, -11)$, B $\equiv (8, -5)$

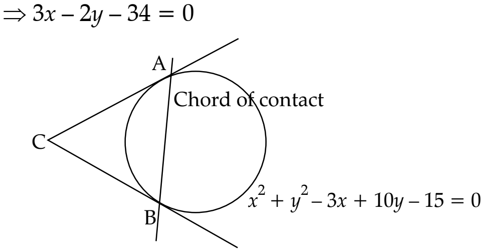
$m_{AB} = \dfrac{-5+11}{8-4} = \dfrac{6}{4} = \dfrac{3}{2}$

Equation of line AB is

$y + 11 = \dfrac{3}{2}(x-4)$

$\Rightarrow 2y + 22 = 3x - 12$

$\Rightarrow 3x - 2y - 34 = 0$ ...(1)



Let C be (α, β)

AB is chord of contact of circle

$x^2 + y^2 - 3x + 10y - 15 = 0$ w.r.t. point C.

So, equation to AB is,

$$\alpha x+\beta y-\frac{3}{2}(x+\alpha)+5(y+\beta)-15=0$$

$$\Rightarrow x\left(\alpha-\frac{3}{2}\right)+y(\beta+5)-\frac{3\alpha}{2}+5\beta-15=0 \quad ...(2)$$

Comparing (1) & (2), we get

$$\frac{\alpha-\frac{3}{2}}{3}=\frac{\beta+5}{-2}=\frac{\frac{-3\alpha}{2}+5\beta-15}{-34}$$

Now, $\frac{\alpha-\frac{3}{2}}{3}=\frac{\beta+5}{-2}$

$\Rightarrow -2\alpha + 3 = 3\beta + 15$

$\Rightarrow 2\alpha + 3\beta + 12 = 0 \quad ...(3)$

Also, $\frac{\alpha-\frac{3}{2}}{3}=\frac{-\frac{3\alpha}{2}+5\beta+15}{-34}$

$$\Rightarrow \frac{2\alpha-3}{6}=\frac{-3\alpha+10\beta-30}{-68}$$

$\Rightarrow -34\,(2\alpha - 3) = 3\,(-3\alpha + 10\beta - 30)$

$\Rightarrow -68\alpha + 102 = -9\alpha + 30\beta - 90$

$\Rightarrow 59\alpha + 30\beta - 192 = 0 \quad ...(4)$

$(4) - (3) \times 10$

$\Rightarrow 39\alpha - 312 = 0 \Rightarrow \alpha = 8$

So, $2(8) + 3\beta + 12 = 0$ [using (3)]

$$\Rightarrow \beta=\frac{-28}{3}$$

Since, AB is tangent to a circle whose centre is $C\left(8,-\frac{28}{3}\right)$

So, $r=\left|\frac{3(8)-2\left(\frac{-28}{3}\right)-34}{\sqrt{9+4}}\right|$

$$\Rightarrow r=\left|\frac{72+56-102}{3\sqrt{13}}\right|=\frac{26}{3\sqrt{13}}$$

$$\Rightarrow r=\frac{2\sqrt{13}}{3}$$

HINT:

(1) Write equation of line joining points A & B using slope-point form.

(2) Write equation of chord of contact AB and compare to obtain coordinates of centre of another circle.

18. Option (2) is correct.

Given, $f(x) = x+\left(\frac{a}{\pi^2-4}\right)\sin x+\left(\frac{b}{\pi^2-4}\right)\cos x \quad ...(1)$

Also, $f(x) = x+\int_0^{\frac{\pi}{2}}\sin(x+y)f(y)dy$

$$\Rightarrow f(x)=x+\int_0^{\frac{\pi}{2}}(\sin x\cos y+\cos x\sin y)f(y)dy$$

$$\Rightarrow f(x)=x+\sin x\int_0^{\frac{\pi}{2}}\cos y f(y)dy + \cos x\int_0^{\frac{\pi}{2}}\sin y\, f(y)dy \quad ...(2)$$

Comparing (1) & (2), we get

$$\frac{a}{\pi^2-4}=\int_0^{\frac{\pi}{2}}\cos y\, f(y)dy \quad ...(3)$$

and $\frac{b}{\pi^2-4}=\int_0^{\frac{\pi}{2}}\sin y\, f(y)dy \quad ...(4)$

Now, (3) + (4)

$$\Rightarrow \frac{a+b}{\pi^2-4}=\int_0^{\frac{\pi}{2}}(\sin y+\cos y)f(y)dy \quad ...(5)$$

Apply $\int_p^q f(x)dx=\int_p^q f(p+q-x)dx$

So, $\frac{a+b}{\pi^2-4}=\int_0^{\frac{\pi}{2}}(\cos y+\sin y)f\left(\frac{\pi}{2}-y\right)dy$

From (1),

$$f\left(\frac{\pi}{2}-y\right)=\left(\frac{\pi}{2}-y\right)+\left(\frac{a}{\pi^2-4}\right)\cos y+\left(\frac{b}{\pi^2-4}\right)\sin y$$

So, $\frac{a+b}{\pi^2-4}=\int_0^{\frac{\pi}{2}}(\cos y+\sin y)$

$$\left\{\left(\frac{\pi}{2}-y\right)+\left(\frac{a}{\pi^2-4}\right)\cos y+\left(\frac{b}{\pi^2-4}\right)\sin y\right\}dy \quad ...(6)$$

Now, (5) + (6)

$$\Rightarrow 2\left(\frac{a+b}{\pi^2-4}\right)=\int_0^{\frac{\pi}{2}}(\sin y+\cos y)\left[\frac{\pi}{2}+\left(\frac{a+b}{\pi^2-4}\right)(\sin y+\cos y)\right]dy$$

$$\Rightarrow 2\left(\frac{a+b}{\pi^2-4}\right)=\frac{\pi}{2}[-\cos y+\sin y]_0^{\frac{\pi}{2}}+\left(\frac{a+b}{\pi^2-4}\right)\int_0^{\frac{\pi}{2}}(\sin y+\cos y)^2dy$$

$$\Rightarrow 2\left(\frac{a+b}{\pi^2-4}\right)=\frac{\pi}{2}[-(0-1)+(1-0)]+\left(\frac{a+b}{\pi^2-4}\right)\int_0^{\frac{\pi}{2}}(1+\sin 2y)dy$$

$$\Rightarrow 2\left(\frac{a+b}{\pi^2-4}\right)=\frac{\pi}{2}(2)+\left(\frac{a+b}{\pi^2-4}\right)\left[y-\frac{\cos 2y}{2}\right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2\left(\frac{a+b}{\pi^2-4}\right)=\pi+\left(\frac{a+b}{\pi^2-4}\right)\left[\left(\frac{\pi}{2}-0\right)-\frac{1}{2}(-1-1)\right]$$

$$\Rightarrow 2\left(\frac{a+b}{\pi^2-4}\right)=\pi+\left(\frac{a+b}{\pi^2-4}\right)\left(\frac{\pi}{2}+1\right)$$

$$\Rightarrow \left(\frac{a+b}{\pi^2-4}\right)\left(1-\frac{\pi}{2}\right)=\pi$$

$\Rightarrow a + b = -2\pi(\pi + 2)$

19. Option (1) is correct.

Given, $f(x) = \dfrac{x^2 + 2x + 1}{x^2 + 1}$

$\Rightarrow f(x) = \dfrac{(x^2+1) + 2x}{x^2+1} \quad \Rightarrow f(x) = 1 + \dfrac{2x}{x^2+1}$

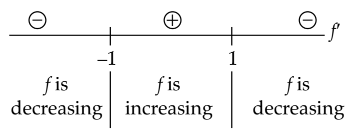
Now, $f'(x) = 0 + \dfrac{2(x^2+1) - 2x(2x)}{(x^2+1)^2}$

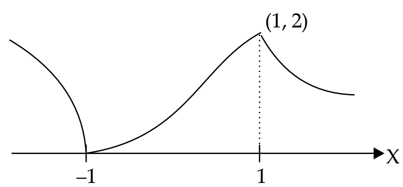
$\Rightarrow f'(x) = \dfrac{2x^2 + 2 - 4x^2}{(x^2+1)^2}$

$\Rightarrow f'(x) = -\dfrac{(x+1)(x-1)}{(x^2+1)^2}$

$f(1) = \dfrac{1+2+1}{1+1} = 2$

$f(-1) = \dfrac{1-2+1}{1+1} = 0$





So, $f(x)$ is one-one in $[1, \infty)$ but not in $(-\infty, \infty)$

20. Option (4) is correct.

Total Apples = 10

Good Apples = 7

Rotten Apples = 3

Let p & q denotes probability of selecting rotten & good apples respectively.

$p = \dfrac{3}{10}$ & $q = \dfrac{7}{10}$

Let X be the number of rotten apples.

$P(X = 0) = {}^4C_0 \left(\dfrac{3}{10}\right)^0 \left(\dfrac{7}{10}\right)\left(\dfrac{6}{9}\right)\left(\dfrac{5}{8}\right)\left(\dfrac{4}{7}\right) = \dfrac{1}{6}$

$P(X = 1) = {}^4C_1 \left(\dfrac{3}{10}\right)^1 \left(\dfrac{7}{9}\right)\left(\dfrac{6}{8}\right)\left(\dfrac{5}{7}\right) = \dfrac{1}{2}$

$P(X = 2) = {}^4C_2 \left(\dfrac{3}{10} \times \dfrac{2}{9}\right)\left(\dfrac{7}{8} \times \dfrac{6}{7}\right) = \dfrac{3}{10}$

$P(X = 3) = {}^4C_3 \left(\dfrac{3}{10} \times \dfrac{2}{9} \times \dfrac{1}{8}\right)\left(\dfrac{7}{7}\right) = \dfrac{1}{30}$

X_i	0	1	2	3
p_i	$\dfrac{1}{6}$	$\dfrac{1}{2}$	$\dfrac{3}{10}$	$\dfrac{1}{30}$

So, $\mu = \displaystyle\sum_{i=0}^{3} p_i x_i$

$= \left(\dfrac{1}{6} \times 0\right) + \left(\dfrac{1}{2} \times 1\right) + \left(\dfrac{3}{10} \times 2\right) + \left(\dfrac{1}{30} \times 3\right)$

$= 0 + \dfrac{1}{2} + \dfrac{3}{5} + \dfrac{1}{10} = \dfrac{5+6+1}{10} = \dfrac{6}{5}$

And $\sigma^2 = \displaystyle\sum_{i=0}^{3} (p_i x_i^2) - \mu^2$

$= \left(\dfrac{1}{6} \times 0^2\right) + \left(\dfrac{1}{2} \times 1^2\right) + \left(\dfrac{3}{10} \times 2^2\right) + \left(\dfrac{1}{30} \times 3^2\right) - \left(\dfrac{6}{5}\right)^2$

$= \dfrac{1}{2} + \dfrac{12}{10} + \dfrac{9}{30} - \dfrac{36}{25}$

$\Rightarrow \sigma^2 = \dfrac{75 + 180 + 45 - 216}{150}$

$\Rightarrow \sigma^2 = \dfrac{84}{150} = \dfrac{14}{25}$

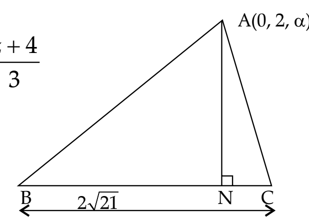
So, $10(\mu^2 + \sigma^2)$

$= 10\left(\dfrac{36}{25} + \dfrac{14}{25}\right) = 10(2) = 20$

Section B

21. Correct answer is [9].

Line $\equiv \dfrac{x+\alpha}{5} = \dfrac{y-1}{2} = \dfrac{z+4}{3}$



Point A ≡ (0, 2, α)

Direction ratios of line BC are (5, 2, 3) or

$5\hat{i} + 2\hat{j} + 3\hat{k}$

Let $N \equiv (-\alpha\hat{i} + \hat{j} - 4\hat{k})$, then

$AN = \dfrac{[\{0 - (-\alpha)\hat{i} + (2-1)\hat{j} + (\alpha+4)\hat{k}\} \times (5\hat{i} + 2\hat{j} + 3\hat{k})]}{|5\hat{i} + 2\hat{j} + 3\hat{k}|}$

$\Rightarrow AN = \dfrac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \alpha & 1 & \alpha+4 \\ 5 & 2 & 3 \end{vmatrix}}{\sqrt{25+4+9}}$

$\Rightarrow AN = \dfrac{|\hat{i}(3 - 2\alpha - 8) - \hat{j}(3\alpha - 5\alpha - 20) + \hat{k}(2\alpha - 5)|}{\sqrt{38}}$

$\Rightarrow AN = \sqrt{\dfrac{(-2\alpha-5)^2 + (-2\alpha-20)^2 + (2\alpha-5)^2}{38}}$

Area of ΔABC = 21

$\Rightarrow \dfrac{1}{2}(BC)(AN) = 21$

$\Rightarrow \dfrac{1}{2}(2\sqrt{21})\sqrt{\dfrac{(2\alpha+5)^2 + (2\alpha+20)^2 + (2\alpha-5)^2}{38}} = 21$

$\Rightarrow (4\alpha^2 + 20\alpha + 25) + (4\alpha^2 + 80\alpha + 400) + (4\alpha^2 - 20\alpha + 25) = 798$

$\Rightarrow 12\alpha^2 + 80\alpha - 348 = 0$

$\Rightarrow 3\alpha^2 + 20\alpha - 87 = 0$

$\Rightarrow (\alpha - 3)(3\alpha + 29) = 0$

But $\alpha \in z \Rightarrow \alpha = 3$

So, $\alpha^2 = 9$

22. Correct answer is [3].

Given, $f(x + y) = f(x) + f(y) - 1, \ \forall x, y \in R$ ...(1)

Put $x = y = 0$

$f(0) = f(0) + f(0) - 1$

$\Rightarrow f(0) = 1$ ...(2)

Consider, $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$\Rightarrow f'(x) = \lim\limits_{h \to 0} \dfrac{(f(x) + f(h) - 1) - f(x)}{h}$, Using (1)

$\Rightarrow f'(x) = \lim\limits_{h \to 0} \dfrac{f(h) - f(0)}{h}$, using (2)

$\Rightarrow f'(x) = f'(0)$

Integrating both sides w.r.t. x, we get

$\int f'(x)dx = \int f'(0)dx$

$\Rightarrow f(x) = xf'(0) + c$

Put $x = 0$

$f(0) = 0 + c$

$\Rightarrow 1 = c$

So, $f(x) = x f'(0) + 1$

$\Rightarrow f(x) = 2x + 1$, as $f'(0) = 2$

Now, $|f(-2)|$

$= |2(-2) + 1| = 3$

23. Correct answer is [10].

Given, $f(x + y) = f(x) + f(y), \forall \ x, y \in \text{N}$

Put $x = 1, y = 1$

$f(2) = 2f(1) = 2\left(\dfrac{1}{5}\right) = \dfrac{2}{5}$

$x = 2, y = 1$

$f(3) = f(2) + f(1) = 3f(1) = \dfrac{3}{5}$

So, generalizing

$f(n) = nf(1) = \dfrac{n}{5}$

Now, $\sum\limits_{n=1}^{m} \dfrac{f(n)}{n(n+1)(n+2)} = \dfrac{1}{12}$

$\Rightarrow \dfrac{1}{5}\sum\limits_{n=1}^{m} \dfrac{1}{(n+1)(n+2)} = \dfrac{1}{12}$

$\Rightarrow \sum\limits_{n=1}^{m} \left(\dfrac{1}{n+1} - \dfrac{1}{n+2}\right) = \dfrac{5}{12}$

$\Rightarrow \left(\dfrac{1}{2} - \dfrac{1}{3}\right) + \left(\dfrac{1}{3} - \dfrac{1}{4}\right) + \ldots + \left(\dfrac{1}{m+1} - \dfrac{1}{m+2}\right) = \dfrac{5}{12}$

$\Rightarrow \dfrac{1}{2} - \dfrac{1}{m+2} = \dfrac{5}{12}$

$\Rightarrow \dfrac{1}{m+2} = \dfrac{1}{12} \Rightarrow m = 10$

HINT:

(1) Put $x = 1, y = 1$; $x = 2, y = 1$ and further generalize to obtain $f(n)$.

(2) Use method of difference to split the general term of series in two parts and further obtian the sum.

24. Correct answer is [1120].

Let three consecutive terms be

$T_{r+1}, T_{r+2}, T_{r+3}$

Now, $\dfrac{T_{r+1}}{T_{r+2}} = \dfrac{2}{5}$

$\Rightarrow \dfrac{{}^nC_r 2^r}{{}^nC_{r+1} 2^{r+1}} = \dfrac{2}{5}$

$\Rightarrow \dfrac{n!}{r!(n-r)!} \cdot \dfrac{(r+1)!(n-1-r)!}{n!} \cdot \dfrac{1}{2} = \dfrac{2}{5}$

$\Rightarrow \dfrac{r+1}{n-r} = \dfrac{4}{5}$ ...(1)

Also, $\dfrac{T_{r+2}}{T_{r+3}} = \dfrac{5}{8}$

$\Rightarrow \dfrac{{}^nC_{r+1} 2^{r+1}}{{}^nC_{r+2} 2^{r+2}} = \dfrac{5}{8}$

$\Rightarrow \dfrac{n!}{(r+1)!(n-r-1)!} \cdot \dfrac{(r+2)!(n-r-2)!}{n!} \cdot \dfrac{1}{2} = \dfrac{5}{8}$

$\Rightarrow \dfrac{r+2}{n-r-1} = \dfrac{5}{4}$ ...(2)

From (1), $n - r = \dfrac{5}{4}(r+1)$

$\Rightarrow 4(r + 2) = 5(n - r) - 5$ [Using (2)]

$\Rightarrow 4r + 13 = 5\left(\dfrac{5}{4}(r+1)\right)$

$\Rightarrow 16r + 52 = 25r + 25$

$\Rightarrow 9r = 27 \Rightarrow r = 3$

Now, $n - 3 = \dfrac{5}{4}(3+1)$

$\Rightarrow 4n - 12 = 20 \Rightarrow n = 8$

So, middle term of these three terms is T_{r+2} i.e., T_5

$T_5 = {}^8C_4(2^4) = \left(\dfrac{8 \times 7 \times 6 \times 5}{24}\right)(16)$

$= (70)(16) = 1120$

25. Correct answer is [60].

Let 'a' & 'r' be the first term and common ratio of G.P. respectively.

$a_4 \times a_6 = 9$

$\Rightarrow ar^3 \cdot ar^5 = 9 \Rightarrow a^2r^8 = 9$

$\Rightarrow ar^4 = -3, 3$

But it's an increasing G.P., so $r > 0$

$\Rightarrow ar^4 = 3, \because$ G.P. has positive numbers ...(1)

$a_5 + a_7 = 24$

$\Rightarrow ar^4 + ar^6 = 24 \Rightarrow ar^6 = 24 - 3$

$\Rightarrow ar^6 = 21$ ...(2)

From (1) & (2)

$r^2 = 7$

$\Rightarrow r = \sqrt{7}$

So, $a = \dfrac{3}{r^4} = \dfrac{3}{49}$

Now, $a_1a_9 + a_2a_4a_9 + a_5 + a_7$

$\Rightarrow a \cdot ar^8 + ar \cdot ar^3 \cdot ar^8 + ar^5 + ar^7$

$= a^2r^8(1 + ar^4) + (a_5 + a_7)$

$= (3)^2 (1 + 3) + 24 = 60$

26. Correct answer is [355].

Given P : $ax + by + 3z = 2(a + b)$

Plane P contains the line L where

$L \equiv x + 10 = \frac{8-y}{2} = z$

or $\frac{x-(-10)}{1} = \frac{y-8}{-2} = \frac{z-0}{1}$

Point on line is (–10, 8, 0), it must lie on plane P.

$\Rightarrow -10a + 8b + 0 = 2a + 2b$

$\Rightarrow 12a - 6b = 0$

$\Rightarrow b = 2a$ ...(1)

Also, normal to plane P is parallel to line L

$\Rightarrow (\hat{i} - 2\hat{j} + \hat{k}).(a\hat{i} + b\hat{j} + 3\hat{k}) = 0$

$\Rightarrow a - 2b + 3 = 0$

$\Rightarrow a - 4a + 3 = 0$ [from (1)]

$\Rightarrow a = 1$

So, $b = 2$

So, $P \equiv x + 2y + 3z - 6 = 0$

Now, perpendicular distance of plane P from point (1, 27, 7) is c.

$$\Rightarrow c = \left|\frac{1.1 + 2.27 + 3.7 - 6}{\sqrt{1^2 + 2^2 + 3^2}}\right|$$

$$\Rightarrow c = \left|\frac{1 + 54 + 21 - 6}{\sqrt{14}}\right| \Rightarrow c = \frac{70}{\sqrt{14}}$$

So, $a^2 + b^2 + c^2$

$$= (1)^2 + (2)^2 + \left(\frac{70}{\sqrt{14}}\right)^2 = 1 + 4 + \frac{70 \times 70}{14} = 355$$

HINT:

If plane $P \equiv ax + by + cz + d = 0$

contains the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$,

then (i) $a\alpha + b\beta + c\gamma + d = 0$

(ii) $(a\hat{i} + b\hat{j} + c\hat{k}).(l\hat{i} + m\hat{j} + n\hat{k}) = 0$

27. Correct answer is [1].

Given $\left(\alpha x^3 + \frac{1}{\beta x}\right)^{11}$

$$\Rightarrow T_{r+1} = {}^{11}C_r (\alpha x^3)^{11-r} \left(\frac{1}{\beta x}\right)^r$$

For coefficient of x^9, put

$\Rightarrow 33 - 3r - r = 9 \Rightarrow 4r = 24 \Rightarrow r = 6$

So, coefficient of x^9 in the expansion of $\left(\alpha x^3 + \frac{1}{\beta x}\right)^{11}$ is

$= {}^{11}C_6 \alpha^5 \beta^{-6}$

Now, in the expansion $\left(\alpha x - \frac{1}{\beta x^3}\right)^{11}$

$$\Rightarrow T_{r+1} = {}^{11}C_r (\alpha x)^{11-r} \left(-\frac{1}{\beta x^3}\right)^r$$

For coefficient of x^{-9}, put $11 - r - 3r = -9 \Rightarrow r = 5$

So, coefficient of x^{-9} in the expansion of $\left(\alpha x - \frac{1}{\beta x^3}\right)^{11}$

$= -{}^{11}C_5 \alpha^6 \beta^{-5}$

Since both coefficients are equal

$\therefore {}^{11}C_6 \alpha^5 \beta^{-6} = -{}^{11}C_5 \alpha^6 \beta^{-5}$

$\Rightarrow \alpha\beta = -1$

So, $(\alpha\beta)^2 = 1$

28. Correct answer is [2].

$A = \vec{a} - \vec{b} + \vec{c}$

$B = \lambda\vec{a} - 3\vec{b} + 4\vec{c}$

$C = -\vec{a} + 2\vec{b} - 3\vec{c}$

$D = 2\vec{a} - 4\vec{b} + 6\vec{c}$

Now, $\overrightarrow{AB} = \vec{a}(\lambda - 1) + \vec{b}(-3 + 1) + \vec{c}(4 - 1)$

$= (\lambda - 1)\vec{a} - 2\vec{b} + 3\vec{c}$

$\overrightarrow{AC} = \vec{a}(-1 - 1) + \vec{b}(2 + 1) + \vec{c}(-3 - 1)$

$= -2\vec{a} + 3\vec{b} - 4\vec{c}$

$\overrightarrow{AD} = \vec{a}(2 - 1) + \vec{b}(-4 + 1) + \vec{c}(6 - 1)$

$= \vec{a} - 3\vec{b} + 5\vec{c}$

Since, $\overrightarrow{AB}, \overrightarrow{AC}$ & $\overrightarrow{AD}$ are coplanar

$\Rightarrow [\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = 0$

$$\Rightarrow \begin{vmatrix} \lambda - 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} \left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right] = 0 \quad \left(\because \left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right] \neq 0\right)$$

$\Rightarrow (\lambda - 1)(15 - 12) + 2(-10 + 4) + 3(6 - 3) = 0$

$\Rightarrow 3\lambda - 3 - 12 + 9 = 0 \Rightarrow \lambda = 2$

29. Correct answer is [1436].

Five digit numbers

Numbers starting with 7

7				
	↓ 5	↓ 5	↓ 5	↓ 5

So, numbers starting with '7' are $5^4 = 625$

5				
	↓ 5	↓ 5	↓ 5	↓ 5

So, number starting with '5' are $5^4 = 625$

3	7			
		↓ 5	↓ 5	↓ 5

Possible numbers = 125

3	5	7		
			↓ 5	↓ 5

= 25 numbers

3	5	5		

= 25 numbers

3	5	3	5	5

= 5 numbers

↓
5

3	5	3	5	

= 5 numbers

↓
5

3	5	3	3	7

$\Rightarrow$ Number is matched,

So, serial number of "35337" is

= 625 + 625 + 125 + 25 + 25 + 5 + 5 + 1 = 1436

30. Correct answer is [32].

Since, all possible numbers are arranged in increasing order.

Also, none of the digit is zero and $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$, where number is

x_1	x_2	x_3	x_4	x_5	x_6

1					

Numbers starting with 1 = ${}^8C_5 = {}^8C_3 = \dfrac{8 \times 7 \times 6}{6} = 56$

2					

Numbers starting with 2 = ${}^7C_5 = 21$

But we are looking for 72th number, so we have to move right, from 2.

2	3				

Possible numbers = ${}^6C_4 = {}^6C_2 = \dfrac{6 \times 5}{2} = 15$

Numbers formed till now = 56 + 15 = 71

So, next number is 72th number = 245678

Sum of digits of 72th number = 2 + 4 + 5 + 6 + 7 + 8 = 32

HINT:

(1) [| | | | |] starting from left most position, form the numbers such that digits are in increasing order & shift position towards right when match occurs.

(2) ${}^nC_r = {}^nC_{n-r}$

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