JEE (Main) MATHEMATICS SOLVED PAPER

2023

29th Jan. Shift 2

Section A

Q. 1. The statement $B \Rightarrow ((\sim A) \vee B)$ is equivalent to:

(1) $A \Rightarrow (A \Leftrightarrow B)$ **(2)** $A \Rightarrow ((\sim A) \Rightarrow B)$

(3) $B \Rightarrow (A \Rightarrow B)$ **(4)** $B \Rightarrow ((\sim A) \Rightarrow B)$

Q. 2. The value of the integral $\int_1^2 \left(\frac{t^4+1}{t^6+1}\right) dt$ is

(1) $\tan^{-1} 2 - \frac{1}{3}\tan^{-1} 8 + \frac{\pi}{3}$

(2) $\tan^{-1} \frac{1}{2} + \frac{1}{3}\tan^{-1} 8 - \frac{\pi}{3}$

(3) $\tan^{-1} \frac{1}{2} - \frac{1}{3}\tan^{-1} 8 + \frac{\pi}{3}$

(4) $\tan^{-1} 2 + \frac{1}{3}\tan^{-1} 8 - \frac{\pi}{3}$

Q. 3. The set of all values of λ for which the equation $\cos^2 2x - 2\sin^4 x - 2\cos^2 x = \lambda$ has a real solution x, is

(1) $[-2, -1]$ **(2)** $\left[-1, -\frac{1}{2}\right]$

(3) $\left[-\frac{3}{2}, -1\right]$ **(4)** $\left[-2, -\frac{3}{2}\right]$

Q. 4. Let R be a relation defined on N as a R b, if $2a + 3b$ is a multiple of 5, $a, b \in$ N. Then R is

(1) an equivalence relation

(2) transitive but not symmetric

(3) not reflexive

(4) symmetric but not transitive

Q. 5. Consider a function f:N $\to$ R, satisfying $f(1) + 2f(2) + 3f(3) + ... + xf(x) = x(x + 1)f(x)$; $x \geq 2$ with $f(1) = 1$.

Then $\frac{1}{f(2022)} + \frac{1}{f(2028)}$ is equal to

(1) 8100 **(2)** 8400 **(3)** 8000 **(4)** 8200

Q. 6. If $\vec{a} = \hat{i} + 2\hat{k}, \vec{b} = \hat{i} + \hat{j} + \hat{k}$, $\vec{c} = 7\hat{i} - 3\hat{j} + 4\hat{k}$, $\vec{r} \times \vec{b} + \vec{b} \times \vec{c} = \vec{0}$ and $\vec{r} \cdot \vec{a} = 0$. Then $\vec{r} \cdot \vec{c}$ is equal to

(1) 32 **(2)** 30 **(3)** 36 **(4)** 34

Q. 7. The shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-6}{-3}$ and $\frac{x-1}{2} = \frac{y+8}{-7} = \frac{z-4}{5}$

(1) $5\sqrt{3}$ **(2)** $2\sqrt{3}$ **(3)** $3\sqrt{3}$ **(4)** $4\sqrt{3}$

Q. 8. The plane $2x - y + z = 4$ intersects the line segment joining the points A(a, -2, 4) and B(2, b, -3) at the point C in the ratio 2 : 1 and the distance of the point C from the origin is $\sqrt{5}$. If $ab < 0$ and P is the point $(a - b, b, 2b - a)$ then CP^2 is equal to

(1) $\frac{97}{3}$ **(2)** $\frac{17}{3}$ **(3)** $\frac{16}{3}$ **(4)** $\frac{73}{3}$

Q. 9. The value of the integral $\int_{\frac{1}{2}}^{2} \frac{\tan^{-1} x}{x} dx$ is equal to

(1) $\frac{\pi}{2}\log_e 2$ **(2)** $\pi \log_3 2$

(3) $\frac{1}{2}\log_e 2$ **(4)** $\frac{1}{4}\log_e 2$

Q. 10. The letters of the word OUGHT are written in all possible ways and these words are arranged as in a dictionary, in a series. Then the serial number of the word TOUGH is

(1) 84 **(2)** 79 **(3)** 89 **(4)** 86

Q. 11. The set of all values of $t \in \mathbb{R}$, for which the matrix

$$\begin{bmatrix} e^t & e^{-t}(\sin t - 2\cos t) & e^{-t}(-2\sin t - \cos t) \\ e^t & e^{-t}(2\sin t + \cos t) & e^{-t}(\sin t - 2\cos t) \\ e^t & e^{-t}\cos t & e^{-t}\sin t \end{bmatrix}$$ is

invertible, is

(1) $\mathbb{R}$ **(2)** $\left\{k\pi + \frac{\pi}{4}, k \in Z\right\}$

(3) $\{k\pi, k \in Z\}$ **(4)** $\left\{(2k+1)\frac{\pi}{2}, k \in Z\right\}$

Q. 12. The area of the region $A = \{(x, y): |\cos x - \sin x| \leq y \leq \sin x, 0 \leq x \leq \frac{\pi}{2}\}$ is

(1) $\sqrt{5} + 2\sqrt{2} - 4.5$ **(2)** $1 - \frac{3}{\sqrt{2}} + \frac{4}{\sqrt{5}}$

(3) $\frac{3}{\sqrt{5}} - \frac{1}{\sqrt{2}} + 1$ **(4)** $\sqrt{5} - 2\sqrt{2} + 1$

Q. 13. The number of 3 digit numbers, that are divisible by either 3 or 4 but not divisible by 48, is

(1) 507 **(2)** 432 **(3)** 472 **(4)** 400

Q. 14. If the lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z+3}{1}$ and $\frac{x-a}{2} = \frac{y+2}{3} = \frac{z-3}{1}$ intersect at the point P, then the distance of the point P from the plane $z = a$ is:

(1) 28 **(2)** 16 **(3)** 10 **(4)** 22

Q. 15. Let $y = y(x)$ be the solution of the differential equation $x\log_e x\frac{dy}{dx} + y = x^2\log_e x$, $(x > 1)$. If $y(2) = 2$, then $y(e)$ is equal to

(1) $\frac{1+e^2}{2}$ (2) $\frac{4+e^2}{4}$ (3) $\frac{2+e^2}{2}$ (4) $\frac{1+e^2}{4}$

Q. 16. Let f and g be twice differentiable function on R such that
$f''(x) = g''(x) + 6x$
$f'(1) = 4g'(1) - 3 = 9$
$f(2) = 3g(2) = 12.$
Then which of the following is NOT true?
(1) There exists $x_0 \in (1, 3/2)$ such that $f(x_0) = g(x_0)$
(2) $|f'(x) - g'(x)| < 6 \Rightarrow -1 < x < 1$
(3) If $-1 < x < 2$, then $|f(x) - g(x)| < 8$
(4) $g(-2) - f(-2) = 20$

Q. 17. If the tangent at a point P on the parabola $y^2 = 3x$ is parallel to the line $x + 2y = 1$ and the tangents at the points Q and R on the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$ are perpendicular to the line $x - y = 2$, then the area of the triangle PQR is :

(1) $\frac{3}{2}\sqrt{5}$ (2) $3\sqrt{5}$ (3) $\frac{9}{\sqrt{5}}$ (4) $5\sqrt{3}$

Q. 18. Let $\vec{a} = 4\hat{i} + 3\hat{j}$ and $\vec{b} = 3\hat{i} - 4\hat{j} + 5\hat{k}$. If $\vec{c}$ is a vector such that $\vec{c}\cdot(\vec{a}\times\vec{b}) + 25 = 0$, $\vec{c}\cdot(\hat{i} + \hat{j} + \hat{k}) = 4$, and projection of $\vec{c}$ on $\vec{a}$ is 1, then the projection of $\vec{c}$ on $\vec{b}$ equals

(1) $\frac{1}{5}$ (2) $\frac{5}{\sqrt{2}}$ (3) $\frac{3}{\sqrt{2}}$ (4) $\frac{1}{\sqrt{2}}$

Q. 19. Let $S = \{w_1, w_2, \ldots\}$ be the sample space associated to a random experiment. Let $P(w_n) = \frac{P(w_{n-1})}{2}, n \geq 2$. Let $A = \{2k + 3l : k, l \in N\}$ and $B = \{w_n : n \in A\}$. Then P(B) is equal to

(1) $\frac{3}{64}$ (2) $\frac{1}{16}$ (3) $\frac{1}{32}$ (4) $\frac{3}{32}$

Q. 20. Let k be the sum of the coefficients of the odd powers of x in the expansion of $(1 + x)^{99}$. Let a be the middle term in the expansion of $\left(2+\frac{1}{\sqrt{2}}\right)^{200}$. If $\frac{{}^{200}C_{99}k}{a} = \frac{2^l m}{n}$, where m and n are odd numbers, then the ordered pair (l, n) is equal to

(1) (50, 51) (2) (50, 101)
(3) (51, 99) (4) (51, 101)

Section B

Q. 21. The total number of 4-digit numbers whose greatest common divisor with 54 is 2 , is

Q. 22. Let $a_1 = b_1 = 1$ and $a_n = a_{n-1} + (n - 1)$, $b_n = b_{n-1} + a_{n-1}$, $\forall n \geq 2$. If $S = \sum_{n=1}^{10}\frac{b_n}{2^n}$ and $T = \sum_{n=1}^{8}\frac{n}{2^{n-1}}$, then $2^7(2S - T)$ is equal to

Q. 23. A triangle is formed by the tangents at the point (2,2) on the curves $y^2 = 2x$ and $x^2 + y^2 = 4x$, and the line $x + y + 2 = 0$. If r is the radius of its circumcircle, then r^2 is equal to

Q. 24. Let $\alpha_1, \alpha_2, \ldots, \alpha_7$ be the roots of the equation $x^7 + 3x^5 - 13x^3 - 15x = 0$ and $|\alpha_1| \geq |\alpha_2| \geq \ldots \geq |\alpha_7|$. Then $\alpha_1\alpha_2 - \alpha_3\alpha_4 + \alpha_5\alpha_6$ is equal to

Q. 25. Let X = {11, 12, 13, … , 40, 41} and Y = {61, 62, 63, …, 90, 91} be the two sets of observations. If $\bar{x}$ and $\bar{y}$ are their respective means and σ^2 is the variance of all the observations in X ∪ Y, then $|\bar{x} + \bar{y} - \sigma^2|$ is equal to

Q. 26. If the equation of the normal to the curve $y = \frac{x-a}{(x+b)(x-2)}$ at the point $(1, -3)$ is $x - 4y = 13$, then the value of $a + b$ is equal to

Q. 27. Let A be a symmetric matrix such that $|A| = 2$ and $\begin{bmatrix} 2 & 1 \\ 3 & \frac{3}{2} \end{bmatrix} A = \begin{bmatrix} 1 & 2 \\ \alpha & \beta \end{bmatrix}$. If the sum of the diagonal elements of A is s, then $\frac{\beta s}{\alpha^2}$ is equal to

Q. 28. Let $\alpha = 8 - 14i$, $A = \left\{z \in C : \frac{\alpha z - \overline{\alpha z}}{z^2 - (\bar{z})^2 - 112i} = 1\right\}$ and $B = \{z \in C : |z + 3i| = 4\}$ Then $\sum_{z \in A \cap B}(\text{Re } z - \text{Im } z)$ is equal to

Q. 29. A circle with centre (2, 3) and radius 4 intersects the line $x + y = 3$ at the points P and Q. If the tangents at P and Q intersect at the point $S(\alpha, \beta)$, then $4\alpha - 7\beta$ is equal to

Q. 30. Let $\{a_k\}$ and $\{b_k\}$, $k \in N$, be two G.P.s with common ratios r_1 and r_2 respectively such that $a_1 = b_1 = 4$ and $r_1 < r_2$. Let $c_k = a_k + b_k$, $k \in N$. If $c_2 = 5$ and $c_3 = \frac{13}{4}$ then $\sum_{k=1}^{\infty} c_k - (12a_6 + 8b_4)$ is equal to

Answer Key

Q. No.	Answer	Topic Name	Chapter Name
1	(2, 3, 4)	Statements and logical operations	Mathematical Reasoning
2	(4)	Integration by Substitution	Definite Integration
3	(3)	Trigonometric Equations	Trigonometry
4	(1)	Types of Relations	Set Theory and Relations
5	(1)	Algebra of Functions	Function
6	(4)	Scalar and Vector Products	Vector Algebra
7	(4)	Skew lines	Three Dimensional Geometry
8	(2)	Plane and a point	Three Dimensional Geometry
9	(1)	Properties of Definite Integration	Definite Integration
10	(3)	Permutations	Permutations and Combinations
11	(1)	Invertible Matrices	Matrices and Determinants
12	(4)	Area Bounded by Curves	Area under Curves
13	(2)	Arithmetic Progressions	Sequences and Series
14	(1)	Plane and a Point	Three Dimensional Geometry
15	(2)	Linear Differential Equations	Differential Equations
16	(3)	Basics of Indefinite Integration	Indefinite Integration
17	(2)	Tangent and Normal	Ellipse
18	(2)	Triple Products	Vector Algebra
19	(1)	Algebra of probability	Probability
20	(2)	Middle term	Binomial Theorem
21	[3000]	Permutations	Permutations and Combinations
22	[461]	Sum upto n terms of special series	Sequences and Series
23	[10]	Circumcentre	Straight lines
24	[9]	Higher degree algebraic equations	Quadratic Equations
25	[603]	Measure of Dispersion	Statistics
26	[4]	Tangent and Normal	Application of Derivatives
27	[5]	Algebra of matrices	Matrices and Determinants
28	[14]	Geometry of a complex number	Complex Numbers
29	[11]	Chord of contact	Circle
30	[9]	Geometric Progressions	Sequences and Series

Solutions

Section A

1. Option (2, 3, 4) is correct.

Given $B \Rightarrow (\sim A) \vee B) \equiv \sim B \vee (\sim A) \vee B$

$\{\because P \Rightarrow Q \equiv \sim P \vee R\}$

$\equiv \sim B \vee B \vee \sim A \equiv T \vee \sim A \equiv T$

Now, $A \Rightarrow (A \Leftrightarrow B) = \sim A \vee (A \Leftrightarrow B)$

$\equiv \sim A \vee [(A \Rightarrow B) \wedge (B \Rightarrow A)]$

$\equiv \sim A \vee [(\sim A \vee B) \wedge (\sim B \vee A)]$

$\equiv [\sim A \vee (\sim A \vee B)] \wedge [\sim A \vee (\sim B \vee A)]$

$\equiv [\sim A \vee B] \wedge [T \vee \sim B] \equiv [A \Rightarrow B] \wedge T \equiv A \Rightarrow B$

Now, $A \Rightarrow (\sim A \Rightarrow B) \equiv A \Rightarrow (A \vee B)$

$\equiv \sim A \vee (A \vee B) = T \vee B \equiv T$

Now, $B \Rightarrow (A \Rightarrow B) \equiv B \Rightarrow (\sim A \vee B)$

$\equiv \sim B \vee (\sim A \vee B) \equiv T$

Now, $B \Rightarrow ((\sim A) \Rightarrow B) \equiv B \Rightarrow (A \vee B)$

$= \sim B \vee (A \vee B) \equiv T$

HINT:

(1) Use $P \Rightarrow Q = \sim P \vee Q$

(2) Use $P \vee T = T$

(3) Use $P \wedge T = P$

(4) Use $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$

2. Option (4) is correct.

Let $A = \int_1^2 \frac{t^4+1}{t^6+1} dt$

$\Rightarrow A = \int_1^2 \frac{t^4+1-t^2+t^2}{t^6+1} dt$

$$\Rightarrow A = \int_1^2 \frac{(t^4 - t^2 + 1) + t^2}{(t^2+1)(t^4 - t^2 + 1)} dt$$

$\{\because (a^3 + b^3) = (a + b)(a^2 + b^2 - ab)\}$

$$\Rightarrow A = \int_1^2 \frac{1}{1+t^2} dt + \int_1^2 \frac{t^2}{t^6+1} dt$$

$$\Rightarrow A = [\tan^{-1} t]_1^2 + \frac{1}{3}\int_1^2 \frac{3t^2}{(t^3)^2+1} dt$$

$$\Rightarrow A = \tan^{-1} 2 - \tan^{-1} 1 + \frac{1}{3}\left[\tan^{-1} t^3\right]_1^2$$

$$\Rightarrow A = \tan^{-1} 2 - \tan^{-1} 1 + \frac{1}{3}\tan^{-1} 8 - \frac{1}{3}\tan^{-1} 1$$

$$\Rightarrow A = \tan^{-1} 2 + \frac{1}{3}\tan^{-1} 8 - \frac{4}{3}\tan^{-1} 1$$

$$\Rightarrow A = \tan^{-1} 2 + \frac{1}{3}\tan^{-1} 8 - \frac{4}{3}\left(\frac{\pi}{4}\right)$$

$$\Rightarrow A = \tan^{-1} 2 + \frac{1}{3}\tan^{-1} 8 - \frac{\pi}{3}$$

3. Option (3) is correct.

Given, $\cos^2 2x - 2\sin^4 x - 2\cos^2 x = \lambda$

$$\Rightarrow \cos^2 2x - 2\left[\frac{1-\cos 2x}{2}\right]^2 - (1 + \cos 2x) = \lambda$$

$$\Rightarrow \cos^2 2x - \frac{1}{2}[1 + \cos^2 2x - 2\cos 2x] - 1 - \cos 2x = \lambda$$

$$\Rightarrow \frac{1}{2}\cos^2 2x - \frac{3}{2} = \lambda$$

$\Rightarrow \cos^2 2x = 2\lambda + 3$

As we know $\cos^2 2x \in [0, 1]$

So, $0 \le 2\lambda + 3 \le 1$

$\Rightarrow -3 \le 2\lambda \le -2$

$$\Rightarrow -\frac{3}{2} \le \lambda \le -1$$

4. Option (1) is correct.

R be a relation defined on N

$a \text{ R } b : 2a + 3b$ is a multiple of 5

For reflexive relation, let $(x, x) \in$ N

$\therefore x \text{ R } x \Rightarrow 2x + 3x$ is a multiple of 5

$\Rightarrow 5x$ is a multiple of 5

$\Rightarrow (x, x) \in$ R

$\Rightarrow$ It is a reflexive relation

For symmetric relation, let $(x, y) \in$ R

$x\text{R}y \Rightarrow 2x + 3y$ is a multiple of 5

$\Rightarrow 2x + 3y = 5\alpha_1, \alpha_1 \in$ N

Now, $5(x + y) = 5\alpha_2$

$\Rightarrow 2x + 3y + 3x + 2y = 5\alpha_2$

$\Rightarrow 3x + 2y = 5\alpha_2 - 5\alpha_1$

$\Rightarrow 2y + 3x = 5(\alpha_2 - \alpha_1)$

$\Rightarrow 2y + 3x$ is a multiple of 5

$\Rightarrow (y, x) \in$ R

$\therefore$ It is a symmetric relation

For transitive relation, let $(x, y) \in$ R, $(y, z) \in$ R

$x\text{R}y \Rightarrow 2x + 3y$ is a multiple of 5

$\Rightarrow 2x + 3y = 5\alpha_1$

$y\text{R}z \Rightarrow 2y + 3z$ is a multiple of 5

$\Rightarrow 2y + 3z = 5\alpha_2$

$\Rightarrow 2x + 5y + 3z = 5\alpha_1 + 5\alpha_2$

$\Rightarrow 2x + 3z = 5(\alpha_1 + \alpha_2 - y)$

$\Rightarrow 2x + 3z$ is a multiple of 5

$\Rightarrow (x, z) \in$ R

$\therefore$ It is a transitive relation.

$\Rightarrow$ R is an equivalence relation.

5. Option (1) is correct.

Given: $f(1) + 2f(2) + 3f(3) + ... + nf(n) = n(n + 1) f(n)$; $n \ge 2$ ...(i)

Replace n by $n + 1$ in above equation, we get

$f(1) + 2 f(2) + 3f(3) + ...+ (n + 1) f(n + 1) = (n + 1)(n + 2) f(n + 1)$ ...(ii)

Equation (ii) – Equation (i), we get

$(n + 1) f(n + 1) = (n + 1)[(n + 2) f(n + 1) - n f(n)]$

$\Rightarrow (n + 1) f(n + 1) - nf(n) = 0$ ...(iii)

Put $n = 2, 3, 4, ... n$ in equation (iii), we get

$3f(3) - 2f(2) = 0$

$4f(4) - 3f(3) = 0$

. . .

. . .

. . .

$(n + 1) f(n + 1) - n f(n) = 0$

Adding all the above equations, we get

$(n + 1) f(n + 1) = 2 f(2)$

$$\Rightarrow f(n+1) = \frac{2f(2)}{n+1} \quad ...(iv)$$

Put $n = 2$ in equation (i), we get

$f(1) + 2f(2) = 6 f(2)$

$$\Rightarrow f(2) = \frac{1}{4} \quad (\because f(1) = 1)$$

From equation (iv), $f(n+1) = \dfrac{1}{2(n+1)}$

$$\Rightarrow f(n) = \frac{1}{2n}$$

So, $\dfrac{1}{f(2022)} + \dfrac{1}{f(2028)} = 4044 + 4056 = 8100$

HINT:

Replace $n \Rightarrow n + 1$ in given equation and solve further.

6. Option (4) is correct.

Given : $\vec{a} = \hat{i} + 2\hat{k}$

$\vec{b} = \hat{i} + \hat{j} + \hat{k}$

$\vec{c} = 7\hat{i} - 3\hat{j} + 4\hat{k}$

$\vec{r} \times \vec{b} + \vec{b} \times \vec{c} = \vec{0}$

$\Rightarrow \vec{r} \times \vec{b} - \vec{c} \times \vec{b} = \vec{0}$ $\quad \{\because \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}\}$

$\Rightarrow (\vec{r} - \vec{c}) \times \vec{b} = \vec{0}$

$\Rightarrow \vec{r} - \vec{c}$ is parallel to $\vec{b}$

$\Rightarrow \vec{r} - \vec{c} = \alpha\vec{b} \quad \Rightarrow \vec{r} = \alpha\vec{b} + \vec{c}$

$\Rightarrow \vec{r} = \alpha(\hat{i} + \hat{j} + \hat{k}) + (7\hat{i} - 3\hat{j} + 4\hat{k})$

$\Rightarrow \vec{r} = (\alpha + 7)\hat{i} + (\alpha - 3)\hat{j} + (\alpha + 4)\hat{k}$

Also, $\vec{r} \cdot \vec{a} = 0$

$\Rightarrow \{(\alpha+7)\hat{i}+(\alpha-3)\hat{j}+(\alpha+4)\hat{k}\}.\{\hat{i}+2\hat{k}\}=0$

$\Rightarrow (\alpha + 7) + 2(\alpha + 4) = 0 \Rightarrow 3\alpha + 15 = 0$

$\Rightarrow \alpha = -5$

$\therefore \vec{r} = (-5+7)\hat{i}+(-5-3)\hat{j}+(-5+4)\hat{k}$

$\Rightarrow \vec{r} = 2\hat{i}-8\hat{j}-\hat{k}$

$\therefore \vec{r}.\vec{c} = (2\hat{i}-8\hat{j}-\hat{k}).(7\hat{i}-3\hat{j}+4\hat{k})$

$= 14 + 24 - 4 = 34$

7. Option (4) is correct.

Given lines are $L_1: \frac{x-1}{2}=\frac{y-2}{1}=\frac{z-6}{-3}$

$L_2: \frac{x-1}{2}=\frac{y+8}{-7}=\frac{z-4}{5}$

then L_1 can be written as

$\vec{r} = (\hat{i}+2\hat{j}+6\hat{k})+\lambda(2\hat{i}+\hat{j}-3\hat{k})$

and L_2 can be written as

$\vec{r} = (\hat{i}-8\hat{j}+4\hat{k})+\mu(2\hat{i}-7\hat{j}+5\hat{k})$

As we know distance between two skew lines $\vec{r}=\vec{a}+\lambda\vec{p}$ and $\vec{r}=\vec{b}+\mu\vec{q}$ is given by

$$D = \left|\frac{(\vec{b}-\vec{a})\cdot(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$$

So, $\vec{b}-\vec{a} = (\hat{i}-8\hat{j}+4\hat{k})-(\hat{i}+2\hat{j}+6\hat{k}) = -10\hat{j}-2\hat{k}$

Now, $\vec{p}\times\vec{q} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & -3\\ 2 & -7 & 5\end{vmatrix}$

$= -16\hat{i}-16\hat{j}-16\hat{k}$

$= -16(\hat{i}+\hat{j}+\hat{k})$

And $|\vec{p}\times\vec{q}| = 16\sqrt{3}$

Now, $(\vec{b}-\vec{a})\cdot(\vec{p}\times\vec{q}) = 160+32 = 192$

So, S.D. $= \left|\frac{192}{16\sqrt{3}}\right| = 4\sqrt{3}$

8. Option (2) is correct.

Given, equation of plane is $2x - y + z = 4$

And points $A(a, -2, 4)$, $B(2, b, -3)$

$\because$ C divides AB in 2 : 1

$\therefore C = \left(\frac{a+4}{3}, \frac{2b-2}{3}, \frac{-2}{3}\right)$

Since, point C lies on given plane.

So, $2\left(\frac{a+4}{3}\right)-\left(\frac{2b-2}{3}\right)-\frac{2}{3}=4$

$\Rightarrow a - b = 2$ ...(i)

Also given that the distance of point C from origin $= \sqrt{5}$

So, $\left(\frac{a+4}{3}\right)^2+\left(\frac{2b-2}{3}\right)^2+\frac{4}{9}=5$

$\Rightarrow \left(\frac{b+6}{3}\right)^2+\left(\frac{2b-2}{3}\right)^2+\frac{4}{9}=5$ [Using (i)]

$\Rightarrow 5b^2 + 4b - 1 = 0$

$\Rightarrow (5b - 1)(b + 1) = 0$

$\Rightarrow b = \frac{1}{5}$ or $b = -1$

For $b = \frac{1}{5}, a = \frac{11}{5}$

For $b = -1$, $a = 1$

$\because ab < 0$

So, $a = 1$, $b = -1$

$\therefore C = \left(\frac{5}{3}, \frac{-4}{3}, \frac{-2}{3}\right)$ and $P = (2, -1, -3)$

Now, $CP^2 = \left(\frac{5}{3}-2\right)^2+\left(\frac{-4}{3}+1\right)^2+\left(-\frac{2}{3}+3\right)^2 = \frac{17}{3}$

9. Option (1) is correct.

Let $I = \int_{\frac{1}{2}}^{2} \frac{\tan^{-1}x}{x}dx$ ...(1)

Put $\frac{1}{x} = \alpha$

$\Rightarrow x = \frac{1}{\alpha} \Rightarrow dx = \frac{-1}{\alpha^2}d\alpha$

$\Rightarrow I = \int_{2}^{\frac{1}{2}} \frac{\tan^{-1}\left(\frac{1}{\alpha}\right)}{\left(\frac{1}{\alpha}\right)}\left(\frac{-1}{\alpha^2}\right)d\alpha$

$\Rightarrow I = \int_{2}^{\frac{1}{2}} \frac{-\cot^{-1}\alpha}{\alpha}d\alpha$ $\left\{\because \tan^{-1}\frac{1}{x} = \cot^{-1}x\right\}$

$\Rightarrow I = \int_{\frac{1}{2}}^{2} \frac{\cot^{-1}\alpha}{\alpha}d\alpha$

$\Rightarrow I = \int_{\frac{1}{2}}^{2} \frac{\cot^{-1}x}{x}dx$ ...(2)

Adding eq. (1) and (2), we get

$2I = \int_{\frac{1}{2}}^{2} \frac{\tan^{-1}x+\cot^{-1}x}{x}dx$

$\Rightarrow 2I = \int_{\frac{1}{2}}^{2} \frac{\frac{\pi}{2}}{x}dx$ $\left\{\because \tan^{-1}x+\cot^{-1}x = \frac{\pi}{2}\right\}$

$\Rightarrow 2I = \frac{\pi}{2}[\ln x]_{\frac{1}{2}}^{2}$

$\Rightarrow 2I = \frac{\pi}{2}\left(\ln 2 - \ln\left(\frac{1}{2}\right)\right)$

$$\Rightarrow I = \frac{\pi}{4}(\ln 2 + \ln 2) \qquad \left\{\because \ln\frac{1}{x} = -\ln x\right\}$$

$$\Rightarrow I = \frac{\pi}{2}\ln 2$$

10. Option (3) is correct.

Given word is OUGHT

In dictionary, the order for letters will be G, H, O, T, U

So, before TOUGH dictionary will have

(1) All words starting from G = 4!

(2) All words starting from H = 4!

(3) All words starting from O = 4!

(4) All words starting from TG = 3!

(5) All words starting from TH = 3!

(6) All words starting from TOG = 2!

(7) All words starting from TOH = 2!

(8) Finally the word TOUGH = 1

So, serial number = (4! × 3) + (3! × 2) + (2! × 2) + 1

= (24 × 3) + (6 × 2) + (4) + 1

= 72 + 12 + 4 + 1 = 89

11. Option (1) is correct.

$$\text{Let } A = \begin{bmatrix} e^t & e^{-t}(\sin t - 2\cos t) & e^{-t}(-2\sin t - \cos t) \\ e^t & e^{-t}(2\sin t + \cos t) & e^{-t}(\sin t - 2\cos t) \\ e^t & e^{-t}\cos t & e^{-t}\sin t \end{bmatrix}$$

$$\Rightarrow |A| = \begin{vmatrix} e^t & e^{-t}(\sin t - 2\cos t) & e^{-t}(-2\sin t - \cos t) \\ e^t & e^{-t}(2\sin t + \cos t) & e^{-t}(\sin t - 2\cos t) \\ e^t & e^{-t}\cos t & e^{-t}\sin t \end{vmatrix}$$

$$\Rightarrow |A| = e^t e^{-t} e^{-t} \begin{vmatrix} 1 & \sin t - 2\cos t & -2\sin t - \cos t \\ 1 & 2\sin t + \cos t & \sin t - 2\cos t \\ 1 & \cos t & \sin t \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$|A| = e^{-t} \begin{vmatrix} 0 & -\sin t - 3\cos t & -3\sin t + \cos t \\ 0 & 2\sin t & -2\cos t \\ 1 & \cos t & \sin t \end{vmatrix}$$

$$\Rightarrow |A| = e^{-t}[2\sin t\cos t + 6\cos^2 t - (-6\sin^2 t + 2\sin t\cos t)]$$

$$\Rightarrow |A| = e^{-t}[6(\sin^2 t + \cos^2 t)]$$

$$\Rightarrow |A| = 6e^{-t} > 0$$

So, A is invertible for $t \in R$

HINT:

If A is an invertible matrix, then $|A| \neq 0$

12. Option (4) is correct.

Given, $A = \{(x, y): |\cos x - \sin x| \leq y \leq \sin x;\ 0 \leq x \leq \frac{\pi}{2}\}$

Lets find intersecting points of $y = |\cos x - \sin x|$ and $y = \sin x$

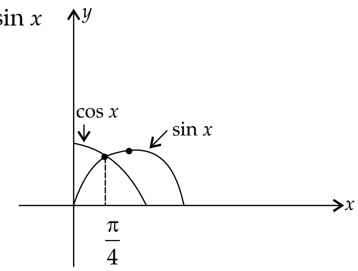
Case 1: When $x \in \left[0, \frac{\pi}{4}\right]$

$|\cos x - \sin x| = \sin x$

$\Rightarrow \cos x - \sin x = \sin x$

$$\Rightarrow \tan x = \frac{1}{2}$$

$$\Rightarrow x = \tan^{-1}\left(\frac{1}{2}\right)$$

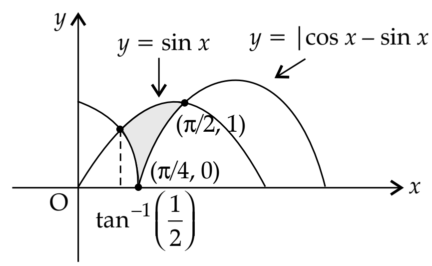


Case 2: When $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

$|\cos x - \sin x| = \sin x$

$\Rightarrow -(\cos x - \sin x) = \sin x$

$$\Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}$$



$$\text{So, } A = \int_{\tan^{-1}\left(\frac{1}{2}\right)}^{\frac{\pi}{4}} [(\sin x) - (\cos x - \sin x)]dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\sin x - (\sin x - \cos x)]dx$$

$$\Rightarrow A = \int_{\tan^{-1}\left(\frac{1}{2}\right)}^{\frac{\pi}{4}} (2\sin x - \cos x)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\, dx$$

$$\Rightarrow A = [-2\cos x - \sin x]_{\tan^{-1}\left(\frac{1}{2}\right)}^{\frac{\pi}{4}} + [\sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$\Rightarrow A = \left[-2\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\right] + \left[2\cos\left(\tan^{-1}\frac{1}{2}\right) + \sin\left(\tan^{-1}\frac{1}{2}\right)\right] + \left[1 - \frac{1}{\sqrt{2}}\right]$$

$$-\frac{3}{\sqrt{2}} + 2\cos\left[\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right] + \sin\left[\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)\right] + 1 - \frac{1}{\sqrt{2}}$$

$$\Rightarrow A = 1 - 2\sqrt{2} + 2\left(\frac{2}{\sqrt{5}}\right) + \frac{1}{\sqrt{5}}$$

$$\Rightarrow A = 1 - 2\sqrt{2} + \sqrt{5}$$

13. Option (2) is correct.

3-digit numbers divisible by 3 are 102, 105, ..., 999 which is an A.P. with first term $a = 102$

common difference $= d = 3$ and last term $l = 999$

$\because l = a(n - 1)d$

$\Rightarrow 999 = 102 + (n-1)\,3$

$\Rightarrow (n-1)\,3 = 897 \Rightarrow n-1 = 299 \Rightarrow n = 300$

So, there are 300 3-digit numbers divisible by 3

Similarly, 3-digit numbers divisible by 4 are 100, 104, ..., 996 which is also an A.P. with $a = 100$, $d = 4$, $l = 996$

$\Rightarrow 996 = 100 + (n-1)\,4$

$\Rightarrow (n-1)\,4 = 896 \Rightarrow n-1 = 224 \Rightarrow n = 225$

So, there are 225 3-digit numbers divisible by 4.

But numbers divisible by 3 as well as 4 are numbers divisible by 12 which are counted in both i.e., 108, 120, ...996 with $a = 108$, $d = 12$, $l = 996$

$\Rightarrow 996 = 108 + (n-1)\,12$

$\Rightarrow (n-1)\,12 = 888$

$\Rightarrow n-1 = 74 \Rightarrow n = 75$

$\therefore$ Total number of 3-digit numbers divisible by 3 or 4 are 300 + 225 – 75 = 450

Now, 3 digit numbers divisible by 48 are 144, 192, ... 960 with $a = 144$, $d = 48$, $l = 960$

$\Rightarrow 960 = 144 + (n-1)\,48$

$\Rightarrow 48(n-1) = 816$

$\Rightarrow n-1 = 17 \Rightarrow n = 18$

$\therefore$ There are 18 3-digit numbers divisible by 48 and these numbers are also divisible 3 and 4

So required numbers are 450 – 18 = 432

14. Option (1) is correct.

Given lines $L_1 : \dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z+3}{1} = m$ (say)

$L_2 : \dfrac{x-a}{2} = \dfrac{y+2}{3} = \dfrac{z-3}{1} = n$ (say)

$\because$ Lines are intersect at P

$\therefore m + 1 = 2n + a$ ...(i)

$2m + 2 = 3n - 2$ ...(ii)

$m - 3 = n + 3$ ...(iii)

On solving equation (i), (ii) and (iii), we get

$m = 22, n = 16, a = -9$

$\therefore P = (m+1, 2m+2, m-3) = (23, 46, 19)$

Now, distance of P (23, 46, 19) from plane $z = -9$ is

$$d = \left|\frac{19+9}{\sqrt{0^2+0^2+1}}\right| = 28$$

15. Option (2) is correct.

Given differential equation is

$x \log_e x \dfrac{dy}{dx} + y = x^2 \log_e x;\ x > 1.$

$\Rightarrow \dfrac{dy}{dx} + \dfrac{1}{x\log_e x} y = x$ which is a linear differential equation

Now, I.F. $= e^{\int \frac{1}{x\log_e x}dx}$

$\Rightarrow$ I.F. $= e^{\log_e(\log_e x)}$

$\Rightarrow$ I.F. $= \log_e x$

So, solution of given differential equation is

$$y(\log_e x) = \int x(\log_e x)dx + C$$

$$\Rightarrow y(\log_e x) = \log_e x\left(\frac{x^2}{2}\right) - \int\left(\frac{1}{x}.\frac{x^2}{2}\right)dx + C$$

$$\Rightarrow y(\log_e x) = \frac{x^2}{2}\log_e x - \frac{1}{2}\left(\frac{x^2}{2}\right) + C$$

$$\Rightarrow y(\log_e x) = \frac{x^2}{2}\log_e x - \frac{1}{4}x^2 + C$$

$\because$ At $x = 2, y = 2$

$\Rightarrow 2(\log_e 2) = 2\log_e 2 - 1 + C$

$\Rightarrow C = 1$

$$\therefore y(\log_e x) = \frac{x^2}{2}\log_e x - \frac{1}{4}x^2 + 1$$

$$\Rightarrow y = \frac{1}{\log_e x}\left[\frac{x^2}{2}\log_e x - \frac{1}{4}x^2 + 1\right]$$

$$\Rightarrow y(e) = \frac{e^2}{2} - \frac{1}{4}e^2 + 1 \Rightarrow y(e) = \frac{4+e^2}{4}$$

16. Option (3) is correct.

Given $f''(x) = g''(x) + 6x$ ...(i)

And $f'(1) = 4g'(1) - 3 = 9$

And $f(2) = 3g(2) = 12$

Integrate equation (i) w.r.t. x, we get

$f'(x) = g'(x) + 3x^2 + C_1$ ...(ii)

Put $x = 1$ in equation (ii), we get

$f'(1) = g'(1) + 3 + C_1$

$\Rightarrow 9 = 3 + 3 + C_1$

$\Rightarrow C_1 = 3$

Integrate equation (ii) w.r.t. x, we get

$f(x) = g(x) + x^3 + 3x + C_2$

Put $x = 2$ in above equation, we get

$f(2) = g(2) + 8 + 6 + C_2$

$\Rightarrow C_2 = -6$

$\therefore f(x) - g(x) = x^3 + 3x - 6$

Let $P(x) = f(x) - g(x) = x^3 + 3x - 6$

Now, $P(1) = -2$ and $P\left(\dfrac{3}{2}\right) = \dfrac{27}{8} + \dfrac{9}{2} - 6 = \dfrac{15}{8}$

$\because P(1)P\left(\dfrac{3}{2}\right) < 0$

$\therefore$ At least one root of $P(x) = 0$ lies in the interval $x \in \left(1, \dfrac{3}{2}\right)$.

So, there exists $x_0 \in \left(1, \dfrac{3}{2}\right)$ such that $f(x_0) = g(x_0)$

Now, $|f'(x) - g'(x)| < 6$

$\Rightarrow 3x^2 + 3 < 6$

$\Rightarrow x^2 + 1 < 2 \Rightarrow x^2 - 1 < 0$

$\Rightarrow (x-1)(x+1) < 0 \Rightarrow x \in (-1, 1)$

Now, $P(-1) = -1 - 3 - 6 = -10$

$P(2) = (2)^3 + 3(2) - 6 = 8$

So, $|P(x)| = |f(x) - g(x)| < 10$

Now, $P(-2) = f(-2) - g(-2) = (-2)^3 + 3(-2) - 6 = -20$

$\Rightarrow g(-2) - f(-2) = 20$

HINT:

(1) Integrate twice the given equation and solve further.

(2) If $f(a)\,f(b) < 0$; then $f(x) = 0$ has at least one root in the interval $x \in (a, b)$.

17. Option (2) is correct.

Given equation of parabola is $y^2 = 3x$

$\because$ Tangent at P on parabola is parallel to the line $x + 2y = 1$

$\therefore$ Slope of tangent $(m) = -\frac{1}{2}$

So, point of contact P $= \left(\frac{a}{m^2}, \frac{2a}{m}\right)$

$$= \left(\frac{\frac{3}{4}}{\frac{1}{4}}, \frac{\frac{3}{2}}{\frac{-1}{2}}\right) = (3, -3)$$

And equation of ellipse is $\frac{x^2}{4} + \frac{y^2}{1} = 1$

$\because$ Tangents at points Q and R are perpendicular to the line $x - y = 2$

So, slope of tangents $(m) = -1$

$\therefore$ Coordinates of points Q, R =

$$\left(\pm\frac{a^2 m}{\sqrt{a^2m^2+b^2}}; \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

$$Q, R = \left(\mp\frac{4}{\sqrt{5}}, \mp\frac{1}{\sqrt{5}}\right)$$

So, $Q = \left(\frac{-4}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$ and $R = \left(\frac{4}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

Now, area of ΔPQR $= \frac{1}{2}\begin{vmatrix} 3 & -3 & 1 \\ \frac{4}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 1 \\ \frac{-4}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 1 \end{vmatrix}$

$$= \frac{1}{2}\begin{vmatrix} 0 & -3 & 1 \\ \frac{5}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 1 \\ -\frac{5}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 1 \end{vmatrix} \quad \{\text{Applying } C_1 \to C_1 + C_2\}$$

$$= \frac{1}{2}\begin{vmatrix} 0 & -3 & 1 \\ 0 & 0 & 2 \\ \frac{-5}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 1 \end{vmatrix} \quad \{\text{Applying } R_2 \to R_2 + R_3\}$$

$$= \frac{1}{2}\left(\frac{-5}{\sqrt{5}}\right)(-6) = 3\sqrt{5}$$

18. Option (2) is correct.

Given, $\vec{a} = 4\hat{i} + 3\hat{j}$ and $\vec{b} = 3\hat{i} - 4\hat{j} + 5\hat{k}$

Let $\vec{c} = p\hat{i} + q\hat{j} + r\hat{k}$

$\because \vec{c}.(\vec{a} \times \vec{b}) + 25 = 0$

$\Rightarrow [\vec{c}\ \vec{a}\ \vec{b}] = -25$

$$\Rightarrow \begin{vmatrix} p & q & r \\ 4 & 3 & 0 \\ 3 & -4 & 5 \end{vmatrix} = -25$$

$\Rightarrow 3p - 4q - 5r = -5$ ...(i)

Also, given that, $\vec{c}.(\hat{i} + \hat{j} + \hat{k}) = 4$

$\Rightarrow p + q + r = 4$ ...(ii)

Also given that projection of $\vec{c}$ on $\vec{a} = 1$

$\Rightarrow \frac{\vec{c}\cdot\vec{a}}{|\vec{a}|} = 1 \Rightarrow \frac{4p + 3q}{5} = 1$

$\Rightarrow 4p + 3q = 5$ ...(iii)

On solving equation (i), (ii) and (iii), we get

$p = 2, q = -1, r = 3$

$\therefore \vec{c} = 2\hat{i} - \hat{j} + 3\hat{k}$

Now, projection on $\vec{c}$ on $\vec{b} = \frac{\vec{c}.\vec{b}}{|\vec{b}|}$

$$= \frac{6 + 4 + 15}{\sqrt{9 + 16 + 25}} = \frac{25}{\sqrt{50}} = \frac{5}{\sqrt{2}}$$

19. Option (1) is correct.

Given : S = $\{w_1, w_2, ...w_n\}$ is the sample space and

$P(w_n) = \frac{P(w_{n-1})}{2}, n \geq 2$

$\Rightarrow P(w_2) = \frac{P(w_1)}{2}$, $P(w_3) = \frac{P(w_2)}{2} = \frac{P(w_1)}{2^2}$,

$P(w_4) = \frac{P(w_3)}{2} = \frac{P(w_1)}{2^3}$

Similarly $P(w_n) = \frac{P(w_1)}{2^{n-1}}$

$\because$ S is the sample space

$\Rightarrow P(w_1) + P(w_2) + ... + P(w_n) = 1$

$\Rightarrow P(w_1) + \frac{P(w_1)}{2} + \frac{P(w_1)}{2^2} + ... = 1$

$\Rightarrow P(w_1)\left\{1 + \frac{1}{2} + \frac{1}{2^2} + ...\right\} = 1$

$\because$ It is an infinite G.P. with $a = 1$ and $r = \frac{1}{2}$

$\Rightarrow P(w_1)\left\{\frac{1}{1 - \frac{1}{2}}\right\} = 1 \Rightarrow P(w_1) = \frac{1}{2}$

Similarly, $P(w_2) = \frac{1}{2}$, $P(w_n) = \frac{1}{2^n}$

Now, A = $\{2k + 3l; k, l \in N\}$

$\Rightarrow$ A = {5, 7, 8, 9, 10, 11, ...}

and B = $\{w_n : n \in A\}$

$\Rightarrow$ B $\{w_5, w_7, w_8, w_9, w_{10}....\}$

$\Rightarrow$ P (B) = $P(w_5) + P(w_7) + P(w_8) + ...$

$\Rightarrow P(B) = \frac{1}{2^5} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + ...$

$\Rightarrow P(B) = \frac{1}{2^5} + \frac{\frac{1}{2^7}}{1 - \frac{1}{2}} \quad \left\{\because \text{In Infinite GP sum} = \frac{a}{1-r}\right\}$

$$\Rightarrow P(B) = \frac{1}{32} + \frac{1}{2^7} \times 2$$

$$\Rightarrow P(B) = \frac{1}{32} + \frac{1}{64} \quad \Rightarrow P(B) = \frac{2+1}{64}$$

$$\Rightarrow P(B) = \frac{3}{64}$$

HINT:

(1) Find $P(w_2)$, $P(w_3)$ ..... $P(w_n)$ in terms of $P(w_1)$ and then use probability of sample space = 1

(2) Sum of an infinite GP with first term = a, and common ratio = r is $\frac{a}{1-r}$.

20. Option (2) is correct.

$(1 + x)^{99} = {}^{99}C_0 + {}^{99}C_1 x + {}^{99}C_2 x^2 + ... + {}^{99}C_{99} x^{99}$

So, $k = {}^{99}C_1 + {}^{99}C_3 + {}^{99}C_5 + ... + {}^{99}C_{99}$

$$\Rightarrow k = \frac{2^{99}}{2} = 2^{98}$$

$$\{\because {}^nC_0 + {}^nC_2 + ... = {}^nC_1 + {}^nC_3 + {}^nC_5 + ... = \frac{2^n}{2}\}$$

Now, a = Middle term in the expansion of $\left(2 + \frac{1}{\sqrt{2}}\right)^{200}$

$$\Rightarrow a = T_{101} = {}^{200}C_{100} 2^{100} \left(\frac{1}{\sqrt{2}}\right)^{100}$$

$$\Rightarrow a = {}^{200}C_{100} 2^{50'}$$

Also given that, $\frac{{}^{200}C_{99} k}{a} = \frac{2^l m}{n}$,

$$\Rightarrow \frac{{}^{200}C_{99} 2^{98}}{{}^{200}C_{100} 2^{50}} = \frac{2^l m}{n}$$

$$\Rightarrow \frac{200!}{99!101!} \times \frac{100!100!}{200!} \times 2^{48} = \frac{2^l m}{n}$$

$$\Rightarrow \frac{100}{101} \times 2^{48} = \frac{2^l m}{n} \Rightarrow \frac{25 \times 2^{50}}{101} = \frac{2^l . m}{n}$$

$$\Rightarrow l = 50, n = 101$$

Section B

21. Correct answer is [3000].

As we know, $54 = 3 \times 3 \times 3 \times 2$

Let k be the 4 digit number such that $gcd\,(54, a) = 2$

$\Rightarrow a$ = All even numbers of 4 digits – even numbers of 4 digits which are multiple of 3.

Now, even numbers of 4 digits = $9 \times 10 \times 10 \times 5$ = 4500

and even number of 4 digits which are multiples of 3 = numbers of 4 digits which are multiples of 6

$$= \frac{9996}{6} - \frac{1002}{6} + 1$$

= 1666 – 167 + 1 = 1500

$\Rightarrow a = 4500 - 1500 \Rightarrow a = 3000$

HINT:

Required numbers = All 4-digit even numbers – 4 digit even numbers which are multiples of 3.

22. Correct answer is [461].

Given : $a_n = a_{n-1} + (n-1)$

$b_n = b_{n-1} + a_{n-1} \; \forall \; n \geq 2$

$$S = \sum_{n=1}^{10} \frac{b_n}{2^n} \text{ and } T = \sum_{n=1}^{8} \frac{n}{2^{n-1}}$$

$$\text{Now, } S = \frac{b_1}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + ... + \frac{b_{10}}{2^{10}} \quad ...(i)$$

$$\Rightarrow \frac{S}{2} = \frac{b_1}{2^2} + \frac{b_2}{2^3} + ... + \frac{b_{10}}{2^{11}} \quad ...(ii)$$

Equation (i) – Equation (ii), we get

$$\frac{S}{2} = \frac{b_1}{2} + \frac{b_2 - b_1}{2^2} + \frac{b_3 - b_2}{2^3} + ... + \frac{b_{10} - b_9}{2^{10}} - \frac{b_{10}}{2^{11}}$$

$$\Rightarrow \frac{S}{2} = \frac{b_1}{2} + \frac{a_1}{2^2} + \frac{a_2}{2^3} + ... + \frac{a_9}{2^{10}} - \frac{b_{10}}{2^{11}} \quad ...(iii)$$

$$\Rightarrow S = b_1 + \frac{a_1}{2} + \frac{a_2}{2^2} + ... + \frac{a_9}{2^9} - \frac{b_{10}}{2^{10}} \quad ...(iv)$$

Equation (iv) – Equation (iii), we get

$$\frac{S}{2} = \frac{b_1}{2} + \frac{a_1}{2} + \frac{a_2 - a_1}{2^2} + \frac{a_3 - a_2}{2^3} + ... + \frac{a_9 - a_8}{2^9} - \frac{a_9}{2^{10}} - \frac{b_{10}}{2^{11}}$$

$$\Rightarrow \frac{S}{2} = \frac{b_1}{2} + \frac{a_1}{2} - \frac{b_{10}}{2^{11}} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + ... + \frac{8}{2^9} - \frac{a_9}{2^{10}}$$

$$\Rightarrow \frac{S}{2} = \frac{1}{2} + \frac{1}{2} - \frac{b_{10}}{2^{11}} - \frac{a_9}{2^{10}} + \frac{1}{2^2}\left[1 + \frac{2}{2} + \frac{3}{2^2} + ... + \frac{8}{2^7}\right]$$

$$\Rightarrow \frac{S}{2} = 1 - \frac{b_{10}}{2^{11}} - \frac{a_9}{2^{10}} + \frac{1}{4}(T)$$

$$\Rightarrow 2S = 4 + T - \frac{b_{10}}{2^9} - \frac{a_9}{2^8}$$

$$\Rightarrow 2S - T = 4 - \frac{a_9}{2^8} - \frac{b_{10}}{2^9}$$

$$\Rightarrow 2^7(2S - T) = 2^9 - \frac{a_9}{2} - \frac{b_{10}}{2^2} \quad ...(v)$$

$\because a_n - a_{n-1} = n - 1$

So, $a_2 - a_1 = 1$

$a_3 - a_2 = 2$

$a_4 - a_3 = 3$

. . .

. . .

. . .

$a_9 - a_8 = 8$

$a_9 - a_1 = 36$

$\Rightarrow a_9 = 37$

Also given, $b_n - b_{n-1} = a_{n-1}$

$\Rightarrow b_{10} - b_1 = a_1 + a_2 + a_3 + ... + a_9$

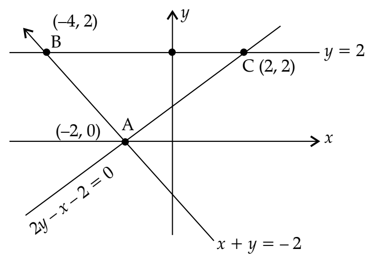
$\Rightarrow b_{10} - 1 = 1 + 2 + 4 + 7 + 11 + 16 + 22 + 29 + 37$

$\Rightarrow b_{10} = 130$

$$\text{So, } 2^7(2S - T) = 2^9 - \frac{37}{2} - \frac{130}{4} = 2^9 - \frac{102}{2} = 461$$

23. Correct answer is [10].

Equation of tangent to the curve $y^2 = 2x$ at point (2, 2) is given by T = 0



$\Rightarrow 2y = 2\left(\frac{x+2}{2}\right)$

$\Rightarrow 2y - x - 2 = 0$ ...(i)

Equation of tangent to the curve $x^2 + y^2 = 4x$ at point (2, 2) is given by T = 0

$\Rightarrow 2x + 2y - 4\left(\frac{x+2}{2}\right) = 0$

$\Rightarrow 2y - 4 = 0$

$\Rightarrow y = 2$

So, triangle is formed by lines $2y - x - 2 = 0$, $y = 2$ and $x + y + 2 = 0$

Now, slope of AB = –1

So, slope of perpendicular bisector of AB = 1

Now, equation of perpendicular bisector of AB is $y - 1 = (x + 3)$

$\Rightarrow y = x + 4$

And equation of perpendicular bisector of BC is $x = -1$

As we know circumcentre is the intersection point of perpendicular bisector of sides of triangle.

$\therefore$ Circumcentre = (–1, 3)

Now, radius $r = \sqrt{(-2+1)^2 + (0-3)^2} = \sqrt{10}$

$\Rightarrow r^2 = 10$

24. Correct answer is [9].

$\Rightarrow$ Given $\alpha_1, \alpha_2, ..., \alpha_7$ are roots of

$x^7 + 3x^5 - 13x^3 - 15x = 0$ and $|\alpha_1| \geq |\alpha_2| \geq ... \geq |\alpha_7|$

Now, $x^7 + 3x^5 - 13x^3 - 15x = 0$

$\Rightarrow x(x^6 + 3x^4 - 13x^2 - 15) = 0$

$\Rightarrow x = 0$ and $x^6 + 3x^4 - 13x^2 - 15 = 0$

$\Rightarrow \alpha_7 = 0$

Now, $x^6 + 3x^4 - 13x^2 - 15 = 0$

Let $y = x^2$

$\Rightarrow y^3 + 3y^2 - 13y - 15 = 0$

$\Rightarrow (y - 3)(y^2 + 6y + 5) = 0$

$\Rightarrow (y - 3)(y + 5)(y + 1) = 0$

$\Rightarrow (x^2 - 3)(x^2 + 5)(x^2 + 1) = 0$

$\Rightarrow x = \pm\sqrt{3}, \pm\sqrt{5}i, \pm i$

$\because |\alpha_1| \geq |\alpha_2| \geq ... \geq |\alpha_7|$

$\Rightarrow |\sqrt{5}i| \geq |-\sqrt{5}i| \geq |\sqrt{3}| \geq |-\sqrt{3}| \geq |i| \geq |-i| \geq 0$

$\Rightarrow \alpha_1 = \sqrt{5}i, \alpha_2 = -\sqrt{5}i, \alpha_3 = \sqrt{3}, \alpha_4 = -\sqrt{3}$,

$\alpha_5 = i, \alpha_6 = -i, \alpha_7 = 0$

$\therefore \alpha_1\alpha_2 - \alpha_3\alpha_4 + \alpha_5\alpha_6$

$= (\sqrt{5}i)(-\sqrt{5}i) - (\sqrt{3})(-\sqrt{3}) + (i)(-i)$

$= -(\sqrt{5}^2)i^2 + 3 - i^2 = 5 + 3 + 1 = 9$

HINT:

Simplify the given equation and find roots using factorisation method and solve it.

25. Correct answer is [603].

Given : X = {11, 12, 13, ... 40, 41}

Y = {61, 62, 63, ... 90, 91}

$\bar{X} = \frac{\Sigma x_i}{n}$

$\Rightarrow \bar{X} = \frac{11+12+13+....+40+41}{31}$

$\Rightarrow \bar{X} = \frac{\frac{31}{2}(11+41)}{31}$

$\left\{\because \text{Sum of AP } = \frac{n}{2}(a+l)\right\}$

$\Rightarrow \bar{X} = \frac{52}{2} = 26$

Similarly, $\bar{Y} = \frac{\Sigma y_i}{n}$

$\Rightarrow \bar{Y} = \frac{61+62+....+90+91}{31}$

$\Rightarrow \bar{Y} = \frac{\frac{31}{2}(61+91)}{31}$ $\left\{\because \text{Sum of AP} = \frac{n}{2}(a+l)\right\}$

$\Rightarrow \bar{Y} = \frac{152}{2} = 76$

Now, σ^2 is variance of all observations in $X \cup Y$

$$\Rightarrow \sigma^2 = \frac{\Sigma x_i^2 + \Sigma y_i^2}{n_x + n_y} - \left(\frac{\Sigma x_i + \Sigma y_i}{n_x + n_y}\right)^2$$

$$\Rightarrow \sigma^2 = \frac{\left\{\sum_{n=1}^{41} n^2 - \sum_{n=1}^{10} n^2\right\} + \left\{\sum_{n=1}^{91} n^2 - \sum_{n=1}^{60} n^2\right\}}{31+31} - \left\{\frac{(31\times 26)+(31\times 76)}{31+31}\right\}^2$$

$$\Rightarrow \sigma^2 = \frac{\left(\frac{41\times 42\times 83}{6} - \frac{10\times 11\times 21}{6}\right) + \left(\frac{91\times 92\times 183}{6} - \frac{60\times 61\times 121}{6}\right)}{62} - \left(\frac{806+2356}{62}\right)^2$$

$$\Rightarrow \sigma^2 = \frac{23821 - 385 + 255346 - 73810}{62} - (51)^2$$

$$\Rightarrow \sigma^2 = \frac{204972}{62} - 2601$$

$\Rightarrow \sigma^2 = 3306 - 2601 \Rightarrow \sigma^2 = 705$

$\therefore\ |\bar{x}+\bar{y}-\sigma^2|=|26+76-705|=603$

26. Correct answer is [04].

Given equation of curve is $y=\dfrac{x-a}{(x+b)(x-2)}$ ...(i)

$\Rightarrow \ln y = \ln(x-a)-\ln(x+b)-\ln(x-2)$

Differentiating the above equation w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{x-a}-\frac{1}{x+b}-\frac{1}{x-2}$$

$$\Rightarrow \frac{dy}{dx}=y\left[\frac{1}{x-a}-\frac{1}{x+b}-\frac{1}{x-2}\right]$$

$$\therefore \left(\frac{dy}{dx}\right)_{(1,-3)}=-3\left[\frac{1}{1-a}-\frac{1}{1+b}+1\right]$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,-3)}=-3\left[\frac{a+b}{(1-a)(1+b)}+1\right]$$

Equation of normal to the given curve at point $(1,-3)$ is $x-4y=13$

So, slope of tangent $=-4$

$$\therefore \left(\frac{dy}{dx}\right)_{(1,-3)}=-3\left[\frac{a+b}{(1-a)(1+b)}+1\right]=-4$$

$$\Rightarrow \frac{-3(a+b)}{(1-a)(1+b)}=-1$$

$\Rightarrow 3(a+b)=(1-a)(1+b)$ ...(ii)

$\because$ Point $(1,-3)$ lies on the curve $y=\dfrac{x-a}{(x+b)(x-2)}$

$$\Rightarrow -3=\frac{1-a}{(1+b)(-1)}$$

$\Rightarrow 3+3b=1-a$

$\Rightarrow a+3b=-2$ ...(iii)

From equation (ii),

$3(-2-3b+b)=(1-(-2-3b))(1+b)$

$\Rightarrow 3(-2-2b)=(3+3b)(1+b)$

$\Rightarrow -6(1+b)=3(1+b)^2$

$\Rightarrow -2=1+b;\ b\neq-1 \Rightarrow b=-3$

Put value of $b=-3$ in equation (iii), we get $a=7$

$\therefore a+b=4$

HINT:

(1) Use slope of tangent to the curve $y=f(x)$ at point (x_1,y_1) is given by $m=\left(\dfrac{dy}{dx}\right)_{(x_1,y_1)}$

27. Correct answer is [05].

Given: A is a symmetric matrix

$$\therefore \begin{bmatrix}2 & 1\\ 3 & \frac{3}{2}\end{bmatrix}A=\begin{bmatrix}1 & 2\\ \alpha & \beta\end{bmatrix}$$

Using property of matrix multiplication, we get order of A is 2×2

$\because$ A is a symmetric matrix of order 2×2

Let A $=\begin{bmatrix}x & y\\ y & z\end{bmatrix}$

where $|A|=2$

$\Rightarrow xz-y^2=2$

and $\begin{bmatrix}2 & 1\\ 3 & \frac{3}{2}\end{bmatrix}\begin{bmatrix}x & y\\ y & z\end{bmatrix}=\begin{bmatrix}1 & 2\\ \alpha & \beta\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}2x+y & 2y+z\\ 3x+\frac{3}{2}y & 3y+\frac{3}{2}z\end{bmatrix}=\begin{bmatrix}1 & 2\\ \alpha & \beta\end{bmatrix}$$

Using equality of matrix, we have

$2x+y=1,\ 2y+z=2,\ 3x+\dfrac{3y}{2}=\alpha$ and $3y+\dfrac{3z}{2}=\beta$

$\Rightarrow y=1-2x,\ z=2-2y$

$\Rightarrow z=2-2+4x$

$\Rightarrow z=4x$

Now, $zx-y^2=2$

$\Rightarrow x(4x)-(1-2x)^2=2$

$\Rightarrow 4x^2-1-4x^2+4x=2$

$\Rightarrow 4x=3 \Rightarrow x=\dfrac{3}{4}$

$$\Rightarrow y=1-2\left(\frac{3}{4}\right)=\frac{-1}{2}\ \Rightarrow z=4\left(\frac{3}{4}\right)=3$$

$$\Rightarrow \alpha=3\left(\frac{3}{4}\right)+\frac{3}{2}\left(\frac{-1}{2}\right)=\frac{3}{2}$$

$$\Rightarrow \beta=3\left(\frac{-1}{2}\right)+\frac{3(3)}{2}=3$$

$$\Rightarrow A=\begin{bmatrix}\frac{3}{4} & \frac{-1}{2}\\ \frac{-1}{2} & 3\end{bmatrix}$$

Sum of diagonal elements of A $=s$

$$\Rightarrow \frac{3}{4}+3=s\Rightarrow s=\frac{15}{4}$$

$$\therefore \frac{\beta s}{\alpha^2}=\frac{3\left(\frac{15}{4}\right)}{\left(\frac{3}{2}\right)^2}=\frac{3\times15}{4\times9}\times4=5$$

28. Correct answer is [14].

Given $\alpha=8-14i$

$$A=\left\{z\in C:\frac{\alpha z-\overline{\alpha z}}{z^2-(\bar{z})^2-112i}=1\right\}$$

$B=\{z\in C:|z+3i|=4\}$

Let $z=x+iy$

Now, $\alpha z-\overline{\alpha z}=(x+iy)(8-14i)-(x-iy)(8+14i)$

$=(16y-28x)i$

Now, $z^2-(\bar{z})^2=(z-\bar{z})(z+\bar{z})$

$=(2yi)(2x)=4xyi$

$$\because \frac{\alpha z-\bar{\alpha}\bar{z}}{z^2-(\bar{z})^2-112i}=1$$

$\Rightarrow \frac{(16y - 28x)i}{4xyi - 112i} = 1$

$\Rightarrow 16y - 28x + 112 = 4xy$

$\Rightarrow 4y - 7x - xy + 28 = 0$

$\Rightarrow 4y - xy - 7x + 28 = 0$

$\Rightarrow y(4 - x) - 7(x - 4) = 0$

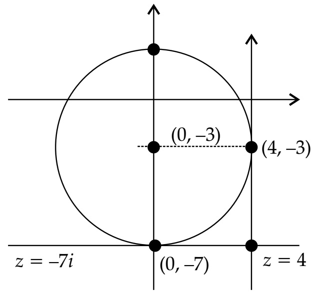
$\Rightarrow (x - 4)(-y - 7) = 0 \Rightarrow x = 4$ or $y = -7$

$\Rightarrow z = 4$ or $z = -7i$

$\because |z + 3i| = 4$

$\Rightarrow (x)^2 + (y + 3)^2 = 16$

So, set B represents all points lies on circle of radius 4 and centre (0, –3)



So, A ∩ B = {(0, –7), (4, –3)}

= {4 – 3i, –7i}

Now, ΣRe (z) – Im (z) = 4 – (–3 –7) = 14

29. Correct answer is [11].

Equation of circle with centre (2, 3) and radius 4 is given by $(x - 2)^2 + (y - 3)^2 = 4^2$

$\Rightarrow x^2 + y^2 - 4x - 6y - 3 = 0$

Equation of line PQ is $x + y = 3$ ...(i)

$\because$ PQ is also the chord of contact w.r.t. S (α, β)

So, equation of chord of contact is T = 0

$\Rightarrow x\alpha + y\beta - 4\left(\frac{x+\alpha}{2}\right) - 6\left(\frac{y+\beta}{2}\right) - 3 = 0$

$\Rightarrow (\alpha - 2)x + (\beta - 3)y - 2\alpha - 3\beta - 3 = 0$ ...(ii)

Comparing equation (i) and equation (ii), we get

$\frac{\alpha - 2}{1} = \frac{\beta - 3}{1} = \frac{-(2\alpha + 3\beta + 3)}{-3}$

So, $\alpha - 2 = \beta - 3$

$\Rightarrow \alpha - \beta = -1$ ...(iii)

And $\alpha - 2 = \frac{2\alpha + 3\beta + 3}{3}$

$\Rightarrow 3\alpha - 6 = 2\alpha + 3\beta + 3$

$\Rightarrow \alpha - 3\beta = 9$ ...(iv)

On solving equation (iii) and equation (iv), we get

$\alpha = -6, \beta = -5$

$\therefore 4\alpha - 7\beta = 4(-6) - 7(-5) = 11$

HINT:

(1) PQ is also the chord of contact w.r.t. S.
(2) Equation of chord of contact is given by T = 0

30. Correct answer is [9].

Given : $\{a_k\}$ is G.P. with common ratio r_1 and $a_1 = 4$

$\Rightarrow$ G.P. $\{a_k\} = 4, 4r_1, 4r_1^2, \ldots$

and $\{b_k\}$ is a G.P. with common ratio r_2 and $b_1 = 4$

$\Rightarrow$ G.P. $\{b_k\} = 4, r_2, 4r_2^2, \ldots$

Now, $c_k = a_k + b_k$

$\Rightarrow c_2 = a_2 + b_2$ and $c_3 = a_3 + b_3$

$\Rightarrow 5 = 4r_1 + 4r_2$ and $\frac{13}{4} = 4r_1^2 + 4r_2^2$

$\Rightarrow r_1 + r_2 = \frac{5}{4}$ and $r_1^2 + r_2^2 = \frac{13}{16}$

$\Rightarrow r_1^2 + r_2^2 + 2r_1r_2 = \frac{25}{16}$

$\Rightarrow 2r_1r_2 = \frac{25}{16} - \frac{13}{16}$

$\Rightarrow r_1r_2 = \frac{12}{16 \times 2} = \frac{3}{8} \Rightarrow r_2 = \frac{3}{8r_1}$

So, $r_1 + \frac{3}{8r_1} = \frac{5}{4}$

$\Rightarrow 8r_1^2 - 10r_1 + 3 = 0$

$\Rightarrow 8r_1^2 - 6r_1 - 4r_1 + 3 = 0$

$\Rightarrow 2r_1(4r_1 - 3) - 1(4r_1 - 3) = 0$

$\Rightarrow (2r_1 - 1)(4r_1 - 3) = 0$

$\Rightarrow r_1 = \frac{1}{2}$ or $\frac{3}{4}$ and $r_2 = \frac{3}{4}, \frac{1}{2}$

$\because r_1 < r_2$

$\therefore r_1 = \frac{1}{2}, r_2 = \frac{3}{4}$

Now, $\sum_{k=1}^{\infty} c_k - (12a_6 + 8b_4)$

$= \sum_{k=1}^{\infty} (a_k + b_k) - \left\{12\left(4\left(\frac{1}{2}\right)^5\right) + 8\left(4\left(\frac{3}{4}\right)^3\right)\right\}$

$= \frac{4}{1 - r_1} + \frac{4}{1 - r_2} - \left(\frac{12}{8} + 8 \times \frac{27}{16}\right)$

$= \frac{4}{1 - \frac{1}{2}} + \frac{4}{1 - \frac{3}{4}} - \left(\frac{3}{2} + \frac{27}{2}\right)$

$= 8 + 16 - 15 = 9$

HINT:

Use n^{th} term of G.P. $= ar^{n-1}$ and sum of infinite G.P. $= \frac{a}{1-r}$

❑❑