JEE (Main) MATHEMATICS SOLVED PAPER

2023
30^{th} Jan. Shift 2

Section A

Q. 1. A vector $\vec{v}$ in the first octant is inclined to the x-axis at 60°, to the y-axis at 45° and to the z-axis at an acute angle. If a plane passing through the points $(\sqrt{2},-1,1)$ and (a, b, c), is normal to $\vec{v}$, then

(1) $\sqrt{2}a+b+c=1$ **(2)** $a+\sqrt{2}b+c=1$
(3) $a+b+\sqrt{2}c=1$ **(4)** $\sqrt{2}a-b+c=1$

Q. 2. Let $a, b, c > 1$, a^3, b^3 and c^3 be in A.P., and $\log_a b$, $\log_c a$ and $\log_b c$ be in G.P. If the sum of first 20 terms of an A.P., whose first term is $\frac{a+4b+c}{3}$ and the common difference is $\frac{a-8b+c}{10}$ is -444, then abc is equal to:

(1) $\frac{125}{8}$ **(2)** 216 **(3)** 343 **(4)** $\frac{343}{8}$

Q. 3. Let $a_1 = 1$, a_2, a_3, a_4, ….. be consecutive natural numbers.

Then $\tan^{-1}\left(\frac{1}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{1}{1+a_2a_3}\right)+....+\tan^{-1}\left(\frac{1}{1+a_{2021}a_{2022}}\right)$

is equal to

(1) $\cot^{-1}(2022)-\frac{\pi}{4}$ **(2)** $\frac{\pi}{4}-\cot^{-1}(2022)$
(3) $\tan^{-1}(2022)-\frac{\pi}{4}$ **(4)** $\frac{\pi}{4}-\tan^{-1}(2022)$

Q. 4. Let $\lambda \in R, \vec{a}=\lambda\hat{i}+2\hat{j}-3\hat{k}, \vec{b}=\hat{i}-\lambda\hat{j}+2\hat{k}$

If $((\vec{a}+\vec{b})\times(\vec{a}\times\vec{b}))\times(\vec{a}-\vec{b})=8\hat{i}-40\hat{j}-24\hat{k}$, then $|\lambda(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|^2$ is equal to

(1) 132 **(2)** 136 **(3)** 140 **(4)** 144

Q. 5. Let q be the maximum integral value of p in [0,10] for which the roots of the equation $x^2 - px + \frac{5}{4}p=0$ are rational. Then the area of the region $\{(x, y): 0 \le y \le (x-q)^2, 0 \le x \le q\}$ is

(1) 243 **(2)** 164 **(3)** $\frac{125}{3}$ **(4)** 25

Q. 6. Let f, g and h be the real valued functions defined on R as

$$f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}, g(x)=\begin{cases} \frac{\sin(x+1)}{(x+1)}, & x \neq -1 \\ 1, & x=-1 \end{cases}$$

and $h(x) = 2[x] - f(x)$, where $[x]$ is the greatest integer $\le x$. Then the value of $\lim_{x\to 1} g(h(x-1))$ is

(1) –1 **(2)** 0 **(3)** sin (1) **(4)** 1

Q. 7. Let S be the set of all values of a_1 for which the mean deviation about the mean of 100 consecutive positive integers $a_1, a_2, a_3, \dots . a_{100}$ is 25. Then S is

(1) N **(2)** ϕ **(3)** {99} **(4)** {9}

Q. 8. For $\alpha, \beta \in \mathbb{R}$, suppose the system of linear equations

$$x - y + z = 5$$
$$2x + 2y + \alpha z = 8$$
$$3x - y + 4z = \beta$$

has infinitely many solutions. Then α and β are the roots of

(1) $x^2 + 14x + 24 = 0$ **(2)** $x^2 + 18x + 56 = 0$
(3) $x^2 - 18x + 56 = 0$ **(4)** $x^2 - 10x + 16 = 0$

Q. 9. Let $\vec{a}$ and $\vec{b}$ be two vectors. Let $|\vec{a}|=1, |\vec{b}| = 4$ and $\vec{a}.\vec{b}=2$ If $\vec{c}=(2\vec{a}\times\vec{b})-3\vec{b}$. then the value of $\vec{b}.\vec{c}$ is

(1) –24 **(2)** –84 **(3)** –48 **(4)** –60

Q. 10. If the functions $f(x) = \frac{x^2}{3}+2bx+\frac{ax^2}{2}$ and $g(x)=\frac{x^2}{3}+ax+bx^2$, $a \neq 2b$ have a common extreme point, then $a + 2b + 7$ is equal to:

(1) $\frac{3}{2}$ **(2)** 3 **(3)** 4 **(4)** 6

Q. 11. If P is a 3×3 real matrix such that $P^T = aP + (a-1)I$, where $a > 1$, then

(1) $|\text{Adj P}| = \frac{1}{2}$

(2) $|\text{Adj P}| = 1$

(3) P is a singular matrix

(4) $|\text{Adj P}| > 1$

Q. 12. The number of ways of selecting two numbers a and b, $a \in \{2,4,6, \dots, 100\}$ and $b \in \{1, 3, 5, \dots, 99\}$ such that 2 is the remainder when $a + b$ is divided by 23 is

(1) 268 **(2)** 108 **(3)** 54 **(4)** 186

Q. 13. $\lim_{n\to\infty}\frac{3}{n}\left\{4+\left(2+\frac{1}{n}\right)^2+\left(2+\frac{2}{n}\right)^2+\dots+\left(3-\frac{1}{n}\right)^2\right\}$ is equal to

(1) 12 **(2)** $\frac{19}{3}$ **(3)** 0 **(4)** 19

Q. 14. Let A be a point on the x-axis. Common tangents are drawn from A to the curves $x^2 + y^2 = 8$ and $y^2 = 16x$. If one of these tangents touches the two curves at Q and R, then $(QR)^2$ is equal to

(1) 81 **(2)** 72 **(3)** 76 **(4)** 64

Q. 15. If a plane passes through the points (–1, k, 0), (2, k, –1), (1, 1, 2) and is parallel to the line $\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$ then the value of $\frac{k^2+1}{(k-1)(k-2)}$ is

(1) $\frac{17}{5}$ **(2)** $\frac{13}{6}$ **(3)** $\frac{6}{13}$ **(4)** $\frac{5}{17}$

Q. 16. The range of the function $f(x) = \sqrt{3-x} + \sqrt{2+x}$ is:

(1) $[2\sqrt{2}, \sqrt{11}]$ **(2)** $[\sqrt{5}, \sqrt{13}]$

(3) $[\sqrt{2}, \sqrt{7}]$ **(4)** $[\sqrt{5}, \sqrt{10}]$

Q. 17. The solution of the differential equation $\frac{dy}{dx} = -\left(\frac{x^2+3y^2}{3x^2+y^2}\right)$, $y(1) = 0$ is

(1) $\log_e|x+y| - \frac{xy}{(x+y)^2} = 0$

(2) $\log_e|x+y| + \frac{2xy}{(x+y)^2} = 0$

(3) $\log_e|x+y| - \frac{2xy}{(x+y)^2} = 0$

(4) $\log_e|x+y| + \frac{xy}{(x+y)^2} = 0$

Q. 18. The parabolas : $ax^2 + 2bx + cy = 0$ and $dx^2 + 2ex + fy = 0$ intersect on the line $y = 1$. If a, b, c, d, e, f are positive real numbers and a, b, c are in G.P., then

(1) d, e, f are in G.P. **(2)** $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P.

(3) d, e, f are in A.P. **(4)** $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in G.P.

Q. 19. Consider the following statements:

P : I have fever

Q: I will not take medicine

R : I will take rest.

The statement "If I have fever, then I will take medicine and I will take rest" is equivalent to:

(1) $((\sim P) \vee \sim Q) \wedge ((\sim P) \vee R)$

(2) $(P \vee Q) \wedge ((\sim P) \vee R)$

(3) $((\sim P) \vee \sim Q) \wedge ((\sim P) \vee \sim R)$

(4) $(P \vee \sim Q) \wedge (P \vee \sim R)$

Q. 20. $x = (8\sqrt{3}+13)^{13}$ and $y = (7\sqrt{2}+9)^9$. If $[t]$ denotes the greatest integer $\le t$, then

(1) $[x]$ is odd but $[y]$ is even

(2) $[x] + [y]$ is even

(3) $[x]$ and $[y]$ are both odd

(4) $[x]$ is even but $[y]$ is odd

Section B

Q. 21. Let a line L pass through the point P(2, 3, 1) and be parallel to the line $x + 3y - 2z - 2 = 0 = x - y + 2z$. If the distance of L from the point (5, 3, 8) is α, then $3\alpha^2$ is equal to ______.

Q. 22. A bag contains six balls of different colours. Two balls are drawn in succession with replacement. The probability that both the balls are of the same colour is p. Next four balls are drawn in succession with replacement and the probability that exactly three balls are of the same colour is q. If $p : q = m : n$, where m and n are coprime, then $m + n$ is equal to ______.

Q. 23. Let P(a_1, b_1) and Q(a_2, b_2) be two distinct points on a circle with center C($\sqrt{2}, \sqrt{3}$). Let O be the origin and OC be perpendicular to both CP and CQ.

If the area of the triangle OCP is $\frac{\sqrt{35}}{2}$, the $a_1^2 + a_2^2 + b_1^2 + b_2^2$ is equal to ______.

Q. 24. Let A be the area of the region $\{(x, y) : y \ge x^2, y \ge (1-x)^2, y \le 2x(1-x)\}$. Then 540 A is equal to ________.

Q. 25. The 8^{th} common term of the series

$S_1 = 3 + 7 + 11 + 15 + 19 + \ldots$

$S_2 = 1 + 6 + 11 + 16 + 21 + \ldots$

is ________.

Q. 26. Let A = {1, 2, 3, 5, 8, 9}. Then the number of possible functions $f : A \to A$ such that $f(m \cdot n) = f(m) \cdot f(n)$ for every $m, n \in A$ with $m \cdot n \in A$ is equal to ______.

Q. 27. If $\int \sqrt{\sec 2x - 1}\, dx = \alpha \log_e \left| \cos 2x + \beta + \sqrt{\cos 2x \left(1 + \cos \frac{1}{\beta} x\right)} \right|$ + constant, then, $\beta - \alpha$ is equal to

Q. 28. If the value of real number $a > 0$ for which $x^2 - 5ax + 1 = 0$ and $x^2 - ax - 5 = 0$ have a common real root is $\frac{3}{\sqrt{2\beta}}$ then β is equal to ______.

Q. 29. 50^{th} root of a number x is 12 and 50^{th} root of another number y is 18 . Then the remainder obtained on dividing $(x + y)$ by 25 is ______.

Q. 30. The number of seven digits odd numbers, that can be formed using all the seven digits 1, 2, 2, 2, 3, 3, 5 is ______.

Answer Key

Q. No.	Answer	Topic Name	Chapter Name
1	(2)	Direction cosines	Three dimensional geometry
2	(2)	G.P.	Sequences and series
3	(2, 3)	Inverse trigonometry	Trigonometry
4	(3)	Vector triple product	Vector algebra
5	(1)	Area between the curves	Integral Calculus
6	(4)	Limit of a Function	Limits
7	(1)	Mean deviation	Statistics
8	(3)	Ssystem of equations	Matrices and determinants
9	(3)	Dot and Cross product	Vector algebra
10	(4)	Extreme Value	Application of derivative
11	(2)	Adjoint	Matrices and determinants
12	(2)	General rule	Permutation and combination
13	(4)	Limit as a sum	Definite Integral
14	(2)	Common tangent	Coordinate geometry
15	(2)	Equation of plane	Three dimensional geometry
16	(4)	Range	Relations and functions
17	(2)	Homogeneous differential equation	Differential equations
18	(2)	Parabola	Conic section
19	(1)	Mathematical statement	Mathematical Reasoning
20	(2)	General rule	Binomial theorem
21	[158]	Distance from point to line	Three dimensional geometry
22	[14]	Basic Probability	Probability
23	[24]	Circle	Conic Section
24	[25]	Area between the curves	Integral calculus
25	[151]	A.P.	Sequences and series
26	[432]	Number of functions	Relations and functions
27	[1]	Indefinite Integral	Integral calculus
28	[13]	Roots of equation	Quadratic equation
29	[23]	Remainder	Binomial theorem
30	[240]	General rule	Permutation and combination

Solutions

Section A

1. Option (2) is correct.

According to question

$l = \cos 60° = \frac{1}{2}, m = \cos 45° = \frac{1}{\sqrt{2}}$

$\because l^2 + m^2 + n^2 = 1 \Rightarrow \frac{1}{4} + \frac{1}{2} + n^2 = 1$

$\Rightarrow \frac{3}{4} + n^2 = 1 \Rightarrow n^2 = 1 - \frac{3}{4} = \frac{1}{4}$

$\Rightarrow n = \frac{1}{2}$ ($\because$ acute angle)

$\therefore \vec{v} = \frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}$

Direction vector of line passes through points $(\sqrt{2}, -1, 1)$ and (a, b, c) is

$(a - \sqrt{2}), (b+1), (c-1)$

$\because$ This line and $\vec{v}$ are perpendicular

$\therefore \frac{1}{2}(a - \sqrt{2}) + \frac{1}{\sqrt{2}}(b+1) + \frac{1}{2}(c-1) = 0$

$\Rightarrow (a - \sqrt{2}) + \sqrt{2}(b+1) + (c-1) = 0$

$\Rightarrow a + \sqrt{2}b + c = 1$

2. Option (2) is correct.

Given that a^3, b^3 and c^3 are in A.P.

$\therefore 2b^3 = a^3 + c^3$ ...(i)

$\log_a b$, $\log_c a$ and $\log_b c$ are in G.P.

$\therefore (\log_c a)^2 = \log_a b \times \log_b c = \log_a c$

$\Rightarrow (\log_c a)^2 = \dfrac{1}{\log_c a} \Rightarrow (\log_c a)^3 = 1$

$\Rightarrow \log_c a = 1 \Rightarrow a = c$

$2b^3 = a^3 + b^3$ [from (i)]

$\Rightarrow a = b = c$

First term $= \dfrac{a+4b+c}{3} = \dfrac{a+4a+a}{3} = 2a$

Common difference $= \dfrac{a-8b+c}{10} = \dfrac{a-8a+a}{10} = \dfrac{-3a}{5}$

$S_{20} = \dfrac{20}{2}\left[2(2a) + (20-1)\left(\dfrac{-3a}{5}\right)\right]$

$\Rightarrow -444 = 10\left[4a - \dfrac{57a}{5}\right] \Rightarrow -444 = 10\left[\dfrac{20a - 57a}{5}\right]$

$\Rightarrow -444 = 10 \times \dfrac{-37a}{5}$

$\Rightarrow -444 = -74a \Rightarrow a = 6$

$\Rightarrow abc = 6 \times 6 \times 6 = 216$

3. Option (2, 3) is correct.

Given that $a_1 = 1, a_2, a_3, a_4$ .... are consecutive natural numbers.

$\tan^{-1}\left(\dfrac{1}{1+a_1a_2}\right) + \tan^{-1}\left(\dfrac{1}{1+a_2a_3}\right) + ... + \tan^{-1}\left(\dfrac{1}{1+a_{2021}a_{2022}}\right)$

$= \tan^{-1}\left(\dfrac{a_2-a_1}{1+a_1a_2}\right) + \tan^{-1}\left(\dfrac{a_3-a_2}{1+a_2a_3}\right) + ... + \tan^{-1}\left(\dfrac{a_{2022}-a_{2021}}{1+a_{2021}.a_{2022}}\right)$

$= \tan^{-1}a_2 - \tan^{-1}a_1 + \tan^{-1}a_3 - \tan^{-1}a_2 + ... + \tan^{-1}a_{2022} - \tan^{-1}a_{2021}$

$= \tan^{-1}a_{2022} - \tan^{-1}a_1 = \tan^{-1}a_{2022} - \tan^{-1}1$

$= \tan^{-1}2022 - \dfrac{\pi}{4} = \dfrac{\pi}{2} - \cot^{-1}2022 - \dfrac{\pi}{4}$

$= \dfrac{\pi}{4} - \cot^{-1}2022$

4. Option (3) is correct.

$[(\vec{a}+\vec{b}) \times (\vec{a} \times \vec{b})] \times (\vec{a} - \vec{b})$

$= [\vec{a} \times (\vec{a} \times \vec{b}) + \vec{b} \times (\vec{a} \times \vec{b})] \times (\vec{a} - \vec{b})$

$= [(\vec{a}.\vec{b})\vec{a} - (\vec{a}.\vec{a})\vec{b} + (\vec{b}.\vec{b})\vec{a} - (\vec{b}.\vec{a})\vec{b}] \times (\vec{a} - \vec{b})$

$= 0 - |\vec{a}|^2 (\vec{b} \times \vec{a}) + 0 - (\vec{b}.\vec{a})(\vec{b} \times \vec{a}) - (\vec{a}.\vec{b})(\vec{a} \times \vec{b}) + 0 - |\vec{b}|^2 (\vec{a} \times \vec{b}) - 0$

$= |\vec{a}|^2 (\vec{a} \times \vec{b}) + (\vec{a}.\vec{b})(\vec{a} \times \vec{b}) - (\vec{a}.\vec{b})(\vec{a} \times \vec{b}) - |\vec{b}|^2 (\vec{a} \times \vec{b})$

$= [|\vec{a}|^2 - |\vec{b}|^2](\vec{a} \times \vec{b})$

$= [(\lambda^2 + 4 + 9) - (1 + \lambda^2 + 4)](\vec{a} \times \vec{b})$

$= (\lambda^2 + 13 - 5 - \lambda^2)(\vec{a} \times \vec{b})$

$= 8(\vec{a} \times \vec{b}) = 8\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \lambda & 2 & -3 \\ 1 & -\lambda & 2 \end{vmatrix}$

$= 8[(4 - 3\lambda)\hat{i} - (2\lambda + 3)\hat{j} + (-\lambda^2 - 2)\hat{k}]$

$= 8[\hat{i} - 5\hat{j} - 3\hat{k}]$

$4 - 3\lambda = 1 \Rightarrow \lambda = 1$

$2\lambda + 3 = 5 \Rightarrow \lambda = 1$

$-\lambda^2 - 2 = -3$

$\lambda = \pm 1$

So, $\lambda = 1$

Now, $|\lambda(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b})|^2 = |(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b})|^2$

$= |0 - \vec{a} \times \vec{b} + \vec{b} \times \vec{a} - 0|^2$

$= |2(\vec{a} \times \vec{b})|^2 = 4|\vec{a} \times \vec{b}|^2 = 4(1 + 25 + 9) = 140$

5. Option (1) is correct.

Given that $x^2 - px + \dfrac{5p}{4} = 0$ have rational roots

$\therefore$ D is a perfect square

$\Rightarrow p^2 - 4(1)\dfrac{5}{4}p = p^2 - 5p = p(p-5)$

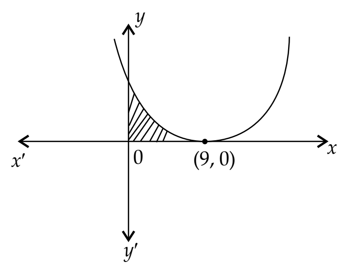
For perfect square

$p = 0, p = 5$ and $p = 9$

Maximum integral value of p is 9.

$q = 9$

$\therefore (x, y); 0 \le y \le (x-9)^2, 0 \le x \le 9\}$



Area $= \int_0^9 (x-9)^2 dx = \left[\dfrac{(x-9)^3}{3}\right]_0^9$

$= \left|0 + \dfrac{9 \times 9 \times 9}{3}\right| = 243$

6. Option (4) is correct.

$\lim_{x \to 1}$ of $g(h(x-1))$

L.H.L. $= \lim_{k \to 0} g(h(1-k-1))$

$= \lim_{k \to 0} g(h(-k)) = \lim_{k \to 0} g(-2+1) = \lim_{k \to 0} g(-1) = 1$

R.H.L. $= \lim_{k \to 0} g(h(k+1)-1))$

$= \lim_{k \to 0} g(h(k)) = \lim_{k \to 0} g(2 \times 0 - 1) = \lim_{k \to 0} g(-1) = 1$

$\because$ L.H.L. = R.H.L.

$\therefore \lim_{x \to 1} g(h(x-1)) = 1$

7. Option (1) is correct.

Let 100 consecutive positive integers

$a_1 = n, a_2 = n + 1, a_3 = n + 1, a_3 = n + 2$,....., $a_{100} = n + 99$

Mean $\bar{x} = \dfrac{n + (n+1) + (n+2) + ... + (n+99)}{100}$

$= \dfrac{100n + \dfrac{99(99+1)}{2}}{100} = \dfrac{100n + \dfrac{100 \times 99}{2}}{100} = n + \dfrac{99}{2}$

Mean deviation about the mean $= \dfrac{\Sigma|x_i - \bar{x}|}{100}$

$$= \frac{1}{100}\left[\frac{99}{2} + \frac{97}{2} + \frac{95}{2} + \ldots + \frac{97}{2} + \frac{99}{2}\right]$$

$$= \frac{2}{100}\left[\frac{99}{2} + \frac{97}{2} + \frac{95}{2} + \ldots \text{. 50 terms}\right]$$

$$= \frac{2}{100} \times \frac{1}{2} \times (50)^2 \text{ [Sum odd natural number is } n^2] = 25$$

So, for all natural number n, mean deviation always 25.

$\therefore$ S = N

8. Option (3) is correct.

Given that system of linear equations has infinitely many solutions.

$$\therefore \begin{vmatrix} 1 & -1 & 1 \\ 2 & 2 & \alpha \\ 3 & -1 & 4 \end{vmatrix} = 0$$

$\Rightarrow 1\,(8 + \alpha) + 1(8 - 3\alpha) + 1\,(-2 - 6) = 0$

$\Rightarrow 8 + \alpha + 8 - 3\alpha - 8 = 0$

$\Rightarrow 8 - 2\alpha = 0 \Rightarrow \alpha = 4$

$$D_Z = \begin{vmatrix} 1 & -1 & 5 \\ 2 & 2 & 8 \\ 3 & -1 & \beta \end{vmatrix} = 0$$

$\Rightarrow 1\,(2\beta + 8) + 1\,(2\beta - 24) + 5\,(-2 - 6) = 0$

$\Rightarrow 2\beta + 8 + 2\beta - 24 - 40 = 0$

$\Rightarrow 4\beta - 56 = 0 \Rightarrow \beta = 14$

Now, quadratic equation whose roots are α, β is

$x^2 - (\alpha + \beta)\,x + \alpha\,.\,\beta = 0$

$\Rightarrow x^2 - (4 + 14)\,x + 4 \times 14 = 0$

$\Rightarrow x^2 - 18x + 56 = 0$

9. Option (3) is correct.

Given that

$\vec{c} = (2\vec{a} \times \vec{b}) - 3\vec{b} \;\Rightarrow \vec{b}.\vec{c} = \vec{b}.(2\vec{a} \times \vec{b}) - 3.\vec{b}.\vec{b}$

$= 0 - 3|\vec{b}|^2 = -3(4)^2 = -48$

10. Option (4) is correct.

$\because f(x) = \frac{x^3}{3} + 2bx + \frac{ax^2}{2}$

$f'(x) = x^2 + 2b + ax = 0$

$g(x) = \frac{x^3}{3} + ax + bx^2$

$g'(x) = x^2 + a + 2bx = 0$

Given that $f(x)$ and $g(x)$ have common extreme point

$\therefore x^2 + 2b + ax = x^2 + a + 2bx = 0$

Since $a \neq 2$ so, it is possible when $x = 1$

$\therefore 1 + 2b + a = 0$

$\Rightarrow a + 2b + 1 + 6 = 6 \Rightarrow a + 2b + 7 = 6$

11. Option (2) is correct.

$\because P^T = aP + (a - 1)\,I$ ...(i)

$\Rightarrow (P^T)^T = (ap)^T + [(a - 1)I]^T$

$\Rightarrow P = aP^T + (a - 1)\,I$ $[\because (P^T)^T = P \text{ and } I^T = I]$

From (i)

$\Rightarrow P = a[aP + (a - 1)I] + (a - 1)\,I$

$\Rightarrow P = a^2P + a\,(a - 1)\,I + (a - 1)I$

$\Rightarrow P - a^2P = (a^2 - a + a - 1)I$

$\Rightarrow P\,(1 - a^2) = -(1 - a^2)\,I \Rightarrow P = -I$

Now, $|\text{Adj } P| = |P|^{3-1} = (-1)^2 = 1$.

12. Option (2) is correct.

$\because a \in \{2, 4, 6, \ldots 100\}$ and $b \in \{1, 3, 5, 77, \ldots 99\}$

So, maximum value of $a + b$ is 199 and minimum value is 3. Sum of odd and even is a always odd.

So, number between 3 and 199 which gives remainder 2 when divided by 23 are 25, 71, 117 and 163

Case 1: Order pair (a, b) such that $a + b = 25$ are (2, 23), (4, 21), (6, 19), (8, 17), (10, 15), (12, 13), (14, 11), (16, 9), (18, 7), (20, 5), (22, 3), (24, 1)

Total ways = 12

Case 2: Order pair (a, b) such that $a + b = 71$ are (2, 69), (4, 67) ...... (70, 1)

Total ways = 35

Case 3: Order pair (a, b) such that $a + b = 117$ are (18, 99), (20, 97), ... (100, 17)

Total ways = 42

Case 4: Order pair (a, b) such that $a + b = 163$ are (64, 99), (66, 97) ..... (100, 63)

Total ways = 19

Hence, total ways = 12 + 35 + 42 + 19 = 108

13. Option (4) is correct.

$$\lim_{n\to\infty} \frac{3}{n}\left\{4 + \left(2 + \frac{1}{n}\right)^2 + \left(2 + \frac{2}{n}\right)^2 + \ldots + \left(3 - \frac{1}{n}\right)^2\right\}$$

$$= \lim_{n\to\infty} \frac{3}{n}\left\{\left(2 + \frac{0}{n}\right)^2 + \left(2 + \frac{1}{n}\right)^2 + \left(2 + \frac{2}{n}\right)^2 + \ldots + \left(2 + \frac{n-1}{n}\right)^2\right\}$$

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{r=0}^{n-1}\left(2 + \frac{r}{n}\right)^2$$

Let $\frac{r}{n} = x$ when $r = 0$ and $n \to \infty \Rightarrow x = 0$

when $r = n - x$ and $n \to \infty \Rightarrow x = 1$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{r=0}^{n-1} 3\left(2 + \frac{r}{n}\right)^2 = \int_0^1 3(2 + x)^2\,dx$$

$$= \left[3\frac{(2 + x)^3}{3}\right]_0^1 = [(2 + x)^3]_0^1$$

$= 27 - 8 = 19$

14. Option (2) is correct.

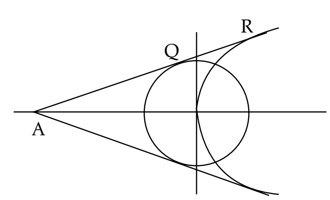
Equation of tangent to the circle $x^2 + y^2 = 8$ is

$y = mx \pm 2\sqrt{2}\sqrt{1 + m^2}$ ...(i)

Equation of tangent to the parabola $y^2 = 16x$ is

$y = mx + \frac{4}{m}$ ...(ii)

Comparing equation (i) and (ii), we get



$\frac{4}{m} = \pm 2\sqrt{2}\sqrt{1 + m^2}$

$\Rightarrow \frac{16}{m^2} = 8(1 + m^2)$

$\Rightarrow 2 = m^4 + m^2$

$\Rightarrow m^4 + m^2 - 2 = 0$

$\Rightarrow m^4 + 2m^2 - m^2 - 2 = 0$

$\Rightarrow m^2\,(m^2 + 2) - 1\,(m^2 + 2) = 0$

$\Rightarrow (m^2 - 1)\,(m^2 + 2) = 0 \Rightarrow m = \pm 1$

$\therefore$ Let $m = 1$

$\therefore y = x + 4$

Contact point on parabola

$$R\left(\frac{4}{m^2}, \frac{8}{m}\right) = R(4, 8)$$

for contanct point on circle solve
$y = x + 4$ and $x^2 + y^2 = 8$
$x^2 + (x + 4)^2 = 8$
$\Rightarrow 2x^2 + 8x + 8 = 0 \Rightarrow x^2 + 4x + 4 = 0$
$\Rightarrow (x + 2)^2 = 0 \Rightarrow x = -2$
$\therefore y = -2 + 4 = 2$
$\therefore$ Q = (–2, 2)
$QR^2 = (\sqrt{(4+2)^2+(8-2)^2})^2 = 36+36 = 72$

15. Option (2) is correct.

Equation of plane passes through the points (–1, *k*, 0), (2, *k*, –1), (1, 1, 2) is

$$\begin{vmatrix} x+1 & y-k & z \\ 2+1 & k-k & -1-0 \\ 1+1 & 1-k & 2-0 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x+1 & y-k & z \\ 3 & 0 & -1 \\ 2 & 1-k & 2 \end{vmatrix} = 0$$

$\Rightarrow (x + 1)[1 - k] - (y - k)(6 + 2) + z(3 - 3k) = 0$
$\Rightarrow (1 - k)x - 8y + (3 - 3k)z + (1 - k + 8k) = 0$
$\Rightarrow (1 - k)x - 8y + (3 - 3k)z + (1 + 7k) = 0$ ...(i)

Given equation of line is

$$\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1} \Rightarrow \frac{x-1}{1} = \frac{y+\frac{1}{2}}{1} = \frac{z+1}{-1} \quad ...(ii)$$

Since plane is parallel to line
$\therefore 1(1 - k) + 1(-8) - 1(3 - 3k) = 0$
$\Rightarrow 1 - k - 8 - 3 + 3k = 0$
$\Rightarrow -10 + 2k = 0 \Rightarrow k = 5$

Now, $\dfrac{k^2+1}{(k-1)(k-2)} = \dfrac{25+1}{4\times 3} = \dfrac{26}{12} = \dfrac{13}{6}$

16. Option (4) is correct.

$f(x) = \sqrt{3-x} + \sqrt{2+x}$

For domain
$3 - x \geq 0 \Rightarrow x \leq 3$
$2 + x \geq 0 \Rightarrow x \geq -2$
Domain = [–2, 3]

$f'(x) = \dfrac{-1}{2\sqrt{3-x}} + \dfrac{1}{2\sqrt{2+x}} = 0$

$\Rightarrow \dfrac{1}{\sqrt{2+x}} = \dfrac{1}{\sqrt{3-x}} \Rightarrow 3 - x = 2 + x \Rightarrow x = \dfrac{1}{2}$

$f(-2) = \sqrt{3+2} + 0 = \sqrt{5}$

$f(3) = 0 + \sqrt{2+3} = \sqrt{5}$

$f\left(\dfrac{1}{2}\right) = \sqrt{3-\dfrac{1}{2}} + \sqrt{2+\dfrac{1}{2}} = \sqrt{\dfrac{5}{2}} + \sqrt{\dfrac{5}{2}} = 2\sqrt{\dfrac{5}{2}} = \sqrt{10}$

$\therefore$ Range = $[\sqrt{5}, \sqrt{10}]$

17. Option (2) is correct.

Given differential equation is

$$\frac{dy}{dx} = -\left(\frac{x^2+3y^2}{3x^2+y^2}\right)$$

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore v + x\frac{dv}{dx} = -\left(\frac{x^2+3v^2x^2}{3x^2+v^2x^2}\right)$$

$$\Rightarrow v + x\frac{dv}{dx} = -\left(\frac{1+3v^2}{3+v^2}\right)$$

$$\Rightarrow \frac{xdv}{dx} = \frac{-1-3v^2-3v-v^3}{3+v^2}$$

$$\Rightarrow \int \frac{3+v^2}{v^3+3v^2+3v+1}dv = -\int \frac{1}{x}dx$$

$$\Rightarrow \int \frac{v^2+3}{(v+1)^3}dv = -\ln x + C$$

Let $v + 1 = t \Rightarrow dv = dt$

$$\int \frac{(t-1)^2+3}{t^3}dt = -\ln x + C$$

$$\Rightarrow \int \frac{t^2-2t+4}{t^3}dt = -\ln x + C$$

$$\Rightarrow \int \left(\frac{1}{t} - \frac{2}{t^2} + \frac{4}{t^3}\right)dt = -\ln x + C$$

$$\Rightarrow \ln t + \frac{2}{t} - \frac{2}{t^2} = -\ln x + C$$

$$\Rightarrow \ln(1+v) + \frac{2}{1+v} - \frac{2}{(1+v)^2} = -\ln x + C$$

$$\Rightarrow \ln\frac{(y+x)}{x} + \frac{2x}{x+y} - \frac{2x^2}{(x+y)^2} = -\ln x + C$$

$$\Rightarrow \ln|x+y| - \ln x + \frac{2x}{x+y} - \frac{2x^2}{(x+y)^2} = -\ln x + C$$

$$\Rightarrow \ln(x+y) + \frac{2x(x+y)-2x^2}{(x+y)^2} = C$$

$$\Rightarrow \ln(x+y) + \frac{2xy}{(x+y)^2} = C$$

Using $y(1) = 0$, we get

$\ln(1 + 0) + \dfrac{0}{(1)^2} = c \Rightarrow c = 0$

So, the solution is:

$$\ln(x+y) + \frac{2xy}{(x+y)^2} = 0$$

18. Option (2) is correct.

Since both curve intersect at $y = 1$
$\therefore ax^2 + 2bx + c = 0$ ...(i)
$dx^2 + 2ex + f = 0$ ...(ii)
Given that a, b, c are in G.P
$\Rightarrow b^2 = ac$ ...(iii)
From equation (i)
$D = 4b^2 - 4ac = 4ac - 4ac = 0$
So, roots are equal.
$\because$ Sum of the roots $= \dfrac{-2b}{a}$

$\Rightarrow \alpha + \alpha = \dfrac{-2b}{a} \Rightarrow \alpha = \dfrac{-b}{a}$

Since, $\alpha = \dfrac{-b}{a}$ is also root of equation (ii)

$$d\left(\frac{-b}{a}\right)^2 + 2e\left(\frac{-b}{a}\right) + f = 0 \Rightarrow d\frac{b^2}{a^2} - \frac{2be}{a} + f = 0$$

$\Rightarrow db^2 - 2abe + a^2f = 0 \Rightarrow db^2 + a^2f = 2abe$

$\Rightarrow dac + a^2f = 2abe \Rightarrow \dfrac{dc}{b^2} + \dfrac{af}{b^2} = \dfrac{2be}{b^2}$

$\Rightarrow \frac{dc}{ac} + \frac{af}{ac} = \frac{2e}{b} \Rightarrow \frac{d}{a} + \frac{f}{c} = \frac{2e}{b}$

$\therefore \frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in AP.

19. Option (1) is correct.

Given that

P : I have fever

Q : I will not take medicine

R : I will take rest

Statement: "If I have fever, then I will take medicine and I will take rest", is

$P \rightarrow (\sim Q \wedge R)$

$\equiv \sim P \vee (\sim Q \wedge R)$

$\equiv (\sim P \vee \sim Q) \wedge (\sim P \vee R)$

20. Option (2) is correct.

$x = (8\sqrt{3}+13)^{13} = {}^{13}C_0(8\sqrt{3})^{13} + {}^{13}C_1(8\sqrt{3})^{12}\, 13^1 + \dots + {}^{13}C_{13}13^{13}$

$x' = (8\sqrt{3}-13)^{13} = {}^{13}C_0(8\sqrt{3})^{13} - {}^{13}C_1(8\sqrt{3})^{12}\, 13^1 + \dots - {}^{13}C_{13}13^{13}$

$\therefore x - x' = 2[{}^{13}C_1(8\sqrt{3})^{12}\, 13^1 + {}^{13}C_3(8\sqrt{3})^{10}.13^3 + \dots + {}^{13}C_{13}13^{13})$

So, $x - x'$ is even integer, hence $[x]$ is even.

$y = (7\sqrt{2}+9)^9 = {}^9C_0(7\sqrt{2})^9 + {}^9C_1(7\sqrt{2})^8.9 + \dots + {}^9C_9 9^9$

$y' = (7\sqrt{2}-9)^9 = {}^9C_0(7\sqrt{2})^9 - {}^9C_1(7\sqrt{2})^8.9 + \dots - {}^9C_9 9^9$

$y - y' = 2[{}^9C_1(7\sqrt{2})^8.9 + {}^9C_3(7\sqrt{2})^6 9^3 + \dots + {}^9C_9 9^9]$

So, $y - y'$ is even integer, hence $[y]$ is even.

$\therefore [x] + [y]$ is even.

Section B

21. Correct answer is [158].

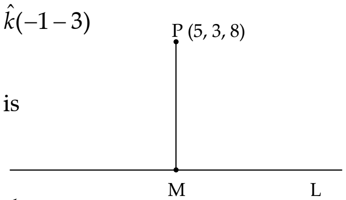
The direction vector $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ 1 & -1 & 2 \end{vmatrix}$

$= \hat{i}(6-2) - \hat{j}(2+2) + \hat{k}(-1-3)$

$= 4\hat{i} - 4\hat{j} - 4\hat{k}$

$\therefore$ Equation of line L is

$\frac{x-2}{4} = \frac{y-3}{-4} = \frac{z-1}{-4}$



$\Rightarrow \frac{x-2}{1} = \frac{y-3}{-1} = \frac{z-1}{-1} = \lambda$

$\Rightarrow x = \lambda + 2, y = -\lambda + 3, z = -\lambda + 1$

$M(\lambda + 2, -\lambda + 3, -\lambda + 1)$

D.r of PM are $<\lambda - 3, -\lambda, -\lambda - 7>$

$\because PM \perp L$

$\therefore 1(\lambda - 3) - 1(-\lambda) - 1(-\lambda - 7) = 0$

$\Rightarrow \lambda - 3 + \lambda + \lambda + 7 = 0$

$\Rightarrow 3\lambda + 4 = 0 \Rightarrow \lambda = -\frac{4}{3}$

$\therefore M\left(\frac{2}{3}, \frac{13}{3}, \frac{7}{3}\right)$

$MP^2 = \left(5 - \frac{2}{3}\right)^2 + \left(3 - \frac{13}{3}\right)^2 + \left(8 - \frac{7}{3}\right)^2$

$\Rightarrow \alpha^2 = \frac{169}{9} + \frac{16}{9} + \frac{289}{9} = \frac{474}{9}$

$\Rightarrow 3\alpha^2 = \frac{3 \times 474}{9} = 158$

22. Correct answer is [14].

$p = {}^6C_1 \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{6}$

$q = {}^6C_1 \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{4!}{3!} = \frac{5}{54}$

$\frac{p}{q} = \frac{\frac{1}{6}}{\frac{5}{54}} = \frac{9}{5} = \frac{m}{n}$

Now, $m + n = 9 + 5 = 14$

23. Correct answer is [24].

Since, OC is perpendicular to both CP and CQ

$\therefore$ PQ is a diameter,

Area of $\Delta OCP = \frac{\sqrt{35}}{2}$

$\Rightarrow \frac{1}{2} CP \times OC = \frac{\sqrt{35}}{2}$

$\Rightarrow CP \times \sqrt{(\sqrt{2})^2 + (\sqrt{3})^2} = \sqrt{35}$

$\Rightarrow CP \times \sqrt{5} = \sqrt{35} \Rightarrow CP = \sqrt{7}$

Radius $= \sqrt{7}$

In ΔOCP

$OP^2 = OC^2 + CP^2$

$\Rightarrow a_1^2 + b_1^2 = 5 + 7 = 12$

Similarly $a_2^2 + b_2^2 = 12$

$\therefore a_1^2 + a_2^2 + b_1^2 + b_2^2 = 12 + 12 = 24$

24. Correct answer is [25].

$y \geq x^2 \Rightarrow y = x^2$ ...(i)

$y \geq (1-x)^2 \Rightarrow y = (1-x)^2$ ...(ii)

$y \leq 2x(1-x) \Rightarrow y = 2x(1-x) = 2x - 2x^2$ ...(iii)

$y = -2\left(x^2 - x + \frac{1}{4} - \frac{1}{4}\right)$

$\Rightarrow y - \frac{1}{2} = -2\left(x - \frac{1}{2}\right)^2$ ...(iii)

from (i) and (ii)

$x^2 = (1-x)^2 \Rightarrow x = \frac{1}{2}$

From (i) and (iii)

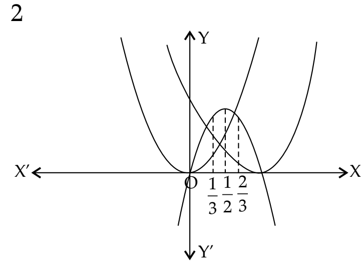
$x^2 = 2x - 2x^2$

$\Rightarrow x = 0, \frac{2}{3}$

From (ii) and (iii)

$(1-x)^2 = 2x - 2x^2$

$x = 1, \frac{1}{3}$



Required area

$$= \int_{\frac{1}{3}}^{\frac{1}{2}} \{(2x - 2x^2) - (1-x)^2\}dx + \int_{\frac{1}{2}}^{\frac{2}{3}} (2x - 2x^2) - x^2\}dx$$

$$= \left[\left(x^2 - \frac{2x^3}{3}\right) + \frac{(1-x)^3}{3}\right]_{\frac{1}{3}}^{\frac{1}{2}} + \left[\left(x^2 - \frac{2x^3}{3}\right) - \frac{x^3}{3}\right]_{\frac{1}{2}}^{\frac{2}{3}}$$

$$A = \frac{30}{24 \times 27}$$

Now, $540A = 540 \times \dfrac{30}{24 \times 27} = 25$

25. Correct answer is [151].

First common term = 11

For common difference of the AP

= L.C.M. of {4, 5} = 20

$\therefore$ A.P : 11, 31, 51 ....

$T_8 = 11 + (8-1)\,20 = 11 + 140 = 151$

26. Correct answer is [432].

$\because f(m.n) = f(m).f(n)$

Put $m = 1$ and $n = 1$

$f(1) = (f(1))^2 \Rightarrow (f(1))^2 - f(1) = 0'$

$\Rightarrow f(1)\,[f(1) - 1] = 0$

$\Rightarrow f(1) = 1$

$f(9) = f(3).f(3) = (f(3))^2$

i.e., $f(9) = 1$ or 3

$\therefore$ Total function = $1 \times 6 \times 2 \times 6 \times 6 \times 1 = 432$

27. Correct answer is [1].

$$I = \int \sqrt{\sec 2x - 1}\,dx = \int \sqrt{\frac{1-\cos 2x}{\cos 2x}}\,dx$$

$$= \int \sqrt{\frac{2\sin^2 x}{2\cos^2 x - 1}}\,dx = \int \frac{\sqrt{2}\sin x}{\sqrt{2\cos^2 x - 1}}\,dx$$

Let $\sqrt{2}\cos x = t \Rightarrow -\sqrt{2}\sin x dx = dt$

$$I = -\int \frac{1}{\sqrt{t^2-1}}dt = -\ln|t + \sqrt{t^2-1}| + C$$

$$= -\ln|\sqrt{2}\cos x + \sqrt{2\cos^2 x - 1}| + C$$

$$= -\ln|\sqrt{2}\cos x + \sqrt{\cos 2x}| + C$$

$$= -\frac{1}{2}\ln(\sqrt{2}\cos x + \sqrt{\cos 2x})^2 + C$$

$$= -\frac{1}{2}\ln|2\cos^2 x + \cos 2x + 2\sqrt{2}\cos x\sqrt{\cos 2x}| + C$$

$$= -\frac{1}{2}\ln|1 + \cos 2x + \cos 2x + 2\sqrt{2}\sqrt{\cos 2x} \times \sqrt{\frac{1+\cos 2x}{2}}| + C$$

$$= -\frac{1}{2}\ln|1 + 2\cos 2x + 2\sqrt{\cos 2x(1+\cos 2x)}| + C$$

$$= \frac{-1}{2}\ln\left|\cos 2x + \frac{1}{2} + \sqrt{\cos 2x(1+\cos 2x)}\right| + C$$

$$= \alpha\ln\left|\cos 2x + \beta + \sqrt{\cos 2x(1 + \cos\frac{1}{\beta}x)}\right| + C$$

$$\therefore \alpha = -\frac{1}{2} \text{ and } \beta = \frac{1}{2}$$

Now, $\beta - \alpha = \dfrac{1}{2} + \dfrac{1}{2} = 1$

28. Correct answer is [13].

$x^2 - 5ax + 1 = x^2 - ax - 5$

$\Rightarrow -5ax + ax = -5 - 1$

$\Rightarrow -4ax = -6 \Rightarrow x = \dfrac{3}{2a}$

$$\therefore \left(\frac{3}{2a}\right)^2 - 5a\left(\frac{3}{2a}\right) + 1 = 0$$

$$\Rightarrow \frac{9}{4a^2} - \frac{15}{2} + 1 = 0$$

$$\Rightarrow \frac{9}{4a^2} = \frac{15}{2} - 1 = \frac{13}{2}$$

$$\Rightarrow a^2 = \frac{2 \times 9}{4 \times 13} = \frac{9}{26}$$

$$\Rightarrow a = \frac{3}{\sqrt{26}} = \frac{3}{\sqrt{2\beta}}$$

$\Rightarrow \beta = 13$

29. Correct answer is [23].

Given that

$x^{\frac{1}{50}} = 12 \Rightarrow x = 12^{50}$

$y^{\frac{1}{50}} = 18 \Rightarrow y = 18^{50}$

$\therefore x + y = 12^{50} + 18^{50}$

$= 6^{50}(2^{50} + 3^{50})$

$= (5+1)^{50}[(2^2)^{25} + (3^2)^{25}]$

$= (25k_1 + 1)[4^{25} + 9^{25}]$

$= (25k_1 + 1)[(5-1)^{25} + (10-1)^{25}]$

$= (25k_1 + 1)[5k_2 - 1 + 10k_3 - 1]$

$= (25k_1 + 1)[25k_2 + 25k_3 - 2]$

$= (25k_1 + 1)[25k - 2]$

$= 625kk_1 - 50k_1 - 25k - 2$

$= 625k_4 - 50k_1 + 25k - 2$

$= 25[25k_4 - 2k_1 + k] - 2$

Remainder = 25 – 2=23.

30. Correct answer is [240].

For odd numbers unit place contain 1, 3, 5.

Case 1: When unit place is 1

Total numbers = $\dfrac{6!}{3!2!} = \dfrac{720}{12} = 60$

Case 2: When unit place is 3

Total numbers = $\dfrac{6!}{3!} = \dfrac{720}{6} = 120$

Case 3: When unit place is 5

Total numbers = $\dfrac{6!}{3!2!} = 60$

Hence total numbers = 60 + 60 + 120 = 240

❑❑