Solved Paper 2014
Mathematics
Class-XII

Time : 3 Hours **Max. Marks : 100**

General Instructions:

(i) All questions are compulsory.

(ii) The question paper consist of 29 questions divided into three sections A, B and C. Section A comprises of 10 questions of one mark each, Section B comprises of 12 questions of four marks each and Section C comprises of 7 questions of six marks each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(iv) There is no overall choice. However, internal choice has been provided in 4 questions of four marks each and 2 questions of six marks each. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculators is not permitted.

Delhi Set I **Code No. 2/1/1**

SECTION - A

*** 1. Let * be binary operation, on the set of all non-zero real numbers, given by $a * b = \frac{ab}{5}$ for all $a, b * R - \{0\}$. Find the value of x, given that $2 * (x * 5) = 10$.** **1**

2. If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then find the value of x. **1**

Sol. $$\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$$

$$\sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1}1$$

$$\sin^{-1}\frac{1}{5} + \cos^{-1}x = \frac{\pi}{2}$$

$$\sin^{-1}\frac{1}{5} = \frac{\pi}{2} - \cos^{-1}x$$

$$\sin^{-1}\frac{1}{5} = \sin^{-1}x$$

$$\therefore \quad x = \frac{1}{5}$$

3. If $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$, find $(x - y)$. **1**

Sol. $$2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 8 \\ 10 & 2x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 8+y \\ 10 & 2x+1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$8 + y = 0$$

$$\Rightarrow \quad y = -8$$

$$2x + 1 = 5$$

$$\Rightarrow \quad x = 2$$

$$x - y = 2 - (-8) = 10$$

4. Solve the following matrix equation for $x : [x \quad 1]\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = 0$. **1**

Sol. $$[x \quad 1]\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = 0$$

$$[x - 2 \quad 0] = 0$$

$$\therefore \quad x - 2 = 0$$

$$\Rightarrow \quad x = 2$$

5. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, write the value of x. **1**

Sol. $$\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$$

$$2x^2 - 40 = 18 + 14$$

$$2x^2 = 72$$

$$x = \sqrt{36} = \pm 6$$

6. Write the antiderivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)$. **1**

Sol. $$= \int\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx$$

$$= 3\int\sqrt{x}dx + \int\frac{1}{\sqrt{x}}dx$$

* *Out of Syllabus*

$$= \frac{3x^{3/2}}{\frac{3}{2}} + \frac{x^{1/2}}{\frac{1}{2}} + C$$

$$= 2x\sqrt{x} + 2\sqrt{x} + C$$

$$= 2\sqrt{x}(x+1) + C$$

7. Evaluate: $\int_0^3 \frac{dx}{9+x^2}$.

Sol.
$$\int_0^3 \frac{dx}{9+x^2} = \frac{1}{3}\left[\tan^{-1}\frac{x}{3}\right]_0^3$$

$$= \frac{1}{3}[\tan^{-1}1 - \tan^{-1}0]$$

$$= \frac{1}{3}\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{12}$$

8. Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $2\hat{i} - 3\hat{j} + 6\hat{k}$. **1**

Sol.
$$\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$$

$$\vec{b} = 2\hat{i} - 3\hat{j} + 6\hat{k}$$

Projection of the vector $\vec{a}$ on $\vec{b}$

$$= \frac{\vec{a}.\vec{b}}{|b|}$$

$$= \frac{(\hat{i} + 3\hat{j} + 7\hat{k}).(2\hat{i} - 3\hat{j} + 6\hat{k})}{|\sqrt{2^2 + (-3)^2 + (6)^2}|}$$

$$= \left|\frac{2 - 9 + 42}{\sqrt{49}}\right|$$

$$= \frac{35}{7} = 5$$

9. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a} + \vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$. **1**

Sol.
$$|\vec{a}| = |\vec{b}| = |\vec{a} + \vec{b}| = 1 \quad \text{(Given)}$$

$$(\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b}$$

$$|(\vec{a} + \vec{b})^2| = |\vec{a}^2| + 2\vec{a}.\vec{b} + |\vec{b}^2|$$

$$1 = 1 + 2|\vec{a}||\vec{b}|\cos\theta + 1$$

$$0 = 2 \times 1 \times 1 \cos\theta + 1$$

$$\cos\theta = -\frac{1}{2}$$

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

*** 10. Write the vector equation of the plane, passing through the point (*a*, *b*, *c*) and parallel to the plane $\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = 2$.** **1**

SECTION - B

11. Let A = {1, 2, 3, ..., 9} and R be relation in A × A defined by (*a*, *b*) R (*c*, *d*) if $a + d = b + c$ for (*a*, *b*), (*c*, *d*) in A × A. Prove that R is an equivalence relation. Also obtain the equivalence class [(2, 5)]. **4**

Sol. $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

R in A × A

$(a, b)\,R(c, d)$ if $(a, b), (c, d) \in A$

$\therefore \quad a + d = b + c$

Consider $(a, b)\,R(a, b)$, $(a, b) \in A \times A$

$a + b = b + a$

Hence, R is reflexive

Consider (a, b) R(c, d) given by $(a, b), (c, d) \in A \times A$

$a + d = b + c$

$\Rightarrow \quad c + b = d + a$

$\Rightarrow (c, d)$ R(a, b)

Hence R is symmetric

Let (a, b) R(c, d) and (c, d) R(e, f)

$(a, b), (c, d), (e, f) \in A \times A$

$a + d = b + c$

and $\quad c + f = d + e$

$a - c = b - d$ ...(i)

$c + f = d + e$ ...(ii)

from (i) & (ii) $a + f = b + e$

$(a, b)\,R(e, f)$

R is transitive

∴ R is an equivalence relation.

$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

a and b such that $2 + b = 5 + a$

$\Rightarrow \quad b = a + 3$

Similarly (2, 5) R(1, 4)

$\Rightarrow \quad 2 + 4 = 5 + 1$

[(2, 5) = (1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)] is the equivalent class under relation R.

12. * Prove that:

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right) = \frac{x}{2};\ x \in \left(0, \frac{\pi}{4}\right)$$ **4**

OR

*** Prove that:**

$$2\tan^{-1}\left(\frac{1}{5}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) + 2\tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}.$$

* *Out of Syllabus*

13. Using properties of determinants, prove that

$$\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} = (x+y+z)^3 \qquad \mathbf{4}$$

Sol. $\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} = (x+y+z)^3$

$$\text{L.H.S.} = \begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$

$C_2 \to C_2 - C_1 \qquad C_3 \to C_3 - C_1$

$$= (x+y+z)\begin{vmatrix} 1 & 0 & 0 \\ 2z & 0 & -(x+y+z) \\ x-y-z & x+y+z & 0 \end{vmatrix}$$

Expanding towards R_1

$$= (x+y+z)\begin{vmatrix} 0 & -(x+y+z) \\ x+y+z & 0 \end{vmatrix}$$

$(x+y+z)^3$ = R.H.S. **Hence Proved**

14. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}(2x\sqrt{1-x^2})$, when $x \neq 0$. **4**

Sol.

$u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ | $V = \cos^{-1}(2x\sqrt{1-x^2})$

Let $x = \sin\theta$

$u = \tan^{-1}\left(\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right)$ | $V = \cos^{-1} |2\sin\theta\sqrt{1-\sin^2\theta}|$

$u = \tan^{-1}\frac{\cos\theta}{\sin\theta}$ | $V = \cos^{-1}(2\sin\theta\cos\theta)$

$u = \tan^{-1}\tan\left(\frac{\pi}{2}-\theta\right)$ | $V = \cos^{-1}\sin 2\theta$

$u = \frac{\pi}{2}-\theta$ | $V = \cos^{-1}\cos\left(\frac{\pi}{2}-2\theta\right)$

$u = \frac{\pi}{2}-\sin^{-1}x$ | $V = \frac{\pi}{2}-2\theta$

$\frac{du}{dx} = 0-\frac{1}{\sqrt{1-x^2}}$ | $V = \frac{\pi}{2}-2\sin^{-1}x$

$$\frac{dV}{dx} = 0-\frac{2}{\sqrt{1-x^2}}$$

$$\frac{du}{dV} = \frac{\frac{du}{dx}}{\frac{dV}{dx}} = \frac{-\frac{1}{\sqrt{1-x^2}}}{-\frac{2}{\sqrt{1-x^2}}} = \frac{1}{2}$$

** Out of Syllabus*

15. If $y = x^x$, prove that $\frac{d^2y}{dx^2}-\frac{1}{y}\left(\frac{dy}{dx}\right)^2-\frac{y}{x}-0.$ **4**

Sol.

$$y = x^x$$

$$\log y = \log x^x = x\log x$$

$$\frac{d}{dx}\log y = \frac{d}{dx}x\log x$$

$$\frac{1}{y}\frac{dy}{dx} = x\times\frac{1}{x}+\log x\times 1$$

$$\frac{1}{y}\frac{dy}{dx} = 1+\log x$$

$$\frac{d}{dx}\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(1+\log x)$$

$$\frac{1}{y}\frac{d^2y}{dx^2}-\frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 = 0+\frac{1}{x}$$

Multiply by y

$$\frac{d^2y}{dx^2}-\frac{1}{y}\left(\frac{dy}{dx}\right)^2-\frac{y}{x} = 0$$ **Hence Proved**

16. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is

(a) Strictly increasing

(b) Strictly decreasing

OR

*** Find the equations of the t tangent and normal to the curve $x = a\sin^3\theta$ and $y = a\cos^3\theta$ at $\theta = \frac{\pi}{4}$.**

Sol.

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

$$f'(x) = 12x^3 - 12x^2 - 24x + 0$$

$$= 12x(x^2 - x - 2)$$

$$= 12x(x-2)(x+1)$$

Critical points 0, –1, and 2

$-\infty$ ⊖ (–1) ⊕ (0) ⊖ (2) ⊕ ∞

$f'(-2) = 12(-2)(-4)(-1) < 0$

$f'(0.5) = 12(-0.5)(-2.5)(0.5) > 0$

$f'(1) = 12(1)(-1)(3) < 0$

$f'(3) = 12(3)(1)(4) > 0$

$\therefore$ Increasing function $(-1, 0) \cup (2, \infty)$

Decreasing function $(-\infty, -1) \cup (0, 2)$

17. Evaluate: $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x.\cos^2 x}dx.$ **4**

OR

Evaluate: $\int (x-3)\sqrt{x^2+3x-18}\,dx.$

Sol. $= \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$

$= \int \frac{(\sin^2 + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x)}{\sin^2 x \cos^2 x} dx$

$= \int \frac{\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} dx$

$= \int (\tan^2 x + \cot^2 x - 1) dx$

$= \int (\sec^2 x + \text{cosec}^2 x - 3) dx$

$= \tan x - \cot x - 3x + C$

OR

$= \int (x+3)\sqrt{x^2+3x-18}\, dx$

$= \int \left(\frac{1}{2}(2x+3) - \frac{9}{2}\right)\sqrt{x^2+3x-18}\, dx$

$= \frac{1}{2}\int (2x+3)\sqrt{x^2+3x-18}\, dx$

$- \frac{9}{2}\int \sqrt{x^2+3x-18}\, dx$

$= \frac{1}{2} \cdot \frac{(x^2+3x-18)^{3/2}}{\frac{3}{2}} - \frac{9}{2}\int \sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\, dx$

$= \frac{1}{3}(x^2+3x-18)^{3/2} - \frac{9}{2}\left[\left(\frac{x+\frac{3}{2}}{2}\right)\sqrt{\left(x^2+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\right.$

$\left. - \frac{81}{8}\log\left(x + \frac{3}{2} + \sqrt{\left(x+\frac{3}{2}\right)^2 - \frac{81}{4}}\right)\right] + C$

$= \frac{1}{3}(x^2+3x-18)^{3/2} - \frac{9}{4}\left(x+\frac{3}{2}\right)\sqrt{(x^2+3x-18)}$

$+ \frac{729}{16}\log\left(x + \frac{3}{2} + \sqrt{(x^2+3x-18)}\right) + C$

18. Find the particular solution of the differential equation $e^x\sqrt{1-y^2}\,dx + \frac{y}{x}dy = 0$ given that $y = 1$ when $x = 0$. **4**

Sol.

$$e^x\sqrt{1-y^2}\,dx + \frac{y}{x}dy = 0$$

$$e^x\sqrt{1-y^2}\,dx = -\frac{y}{x}dy$$

$$xe^x dx = -\frac{y}{\sqrt{1-y^2}}dy$$

$$\int xe^x dx = \int \frac{-y}{\sqrt{1-y^2}}dy$$

$$xe^x - e^x = \frac{1}{2}.2\sqrt{1-y^2} + C$$

$$xe^x - e^x = \sqrt{1-y^2} + C$$

when $x = 0, y = 1$

$$-1 = C$$

$\therefore$ Particular solution is

$$e^x(x-1) = \sqrt{1-y^2} - 1$$

19. Solve the following differential equation:

$$(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$$ **4**

Sol.

$$(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$$

$$\frac{dy}{dx} + \frac{2x}{x^2-1}y = \frac{2}{(x^2-1)^2}$$

Difference equ. is the form of $\frac{dy}{dx} + Py = Q(x)$

$\therefore$ $\text{I.F.} = e^{\int P dx}$

$= e^{\int \frac{2x}{x^2-1}dx}$

$= e^{\log(x^2-1)} = x^2 - 1$

Solution of difference equ. is

$$y \times \text{I.F.} = \int \text{I.F.} \times Q(x) dx$$

$$y(x^2-1) = \int (x^2-1)\frac{2}{(x^2-1)^2}dx$$

$$= 2.\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C$$

$$y(x^2-1) = \log\left|\frac{x-1}{x+1}\right| + C$$

20. Prove that, for any three vectors: $\vec{a}, \vec{b}, \vec{c}$

$$[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] = 2[\vec{a}, \vec{b}, \vec{c}]$$ **4**

OR

Vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are such that $\vec{a}+\vec{b}+\vec{c} = 0$ and $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$. Find the angle between $\vec{a}$ and $\vec{b}$.

Sol. **OR**

$$\vec{a}+\vec{b}+\vec{c} = 0$$

$$\vec{a}+\vec{b} = -\vec{c}$$

$$(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = -\vec{c}(-\vec{c})$$

$$\vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = +|c^2|$$

$$|\vec{a}| + 2\vec{a}.\vec{b} + |\vec{b}|^2 = +|c^2| \quad [a : b = b : a]$$

$$9 + 2\vec{a}.\vec{b} + 25 = +49$$

$$2\vec{a}.\vec{b} = +49 - 34$$
$$2|a||b|\cos\theta = 15$$
$$2.5.3.\cos\theta = 15$$
$$\cos\theta = \frac{15}{30} = \frac{1}{2}$$
$$\theta = 60° \text{ or } \frac{\pi}{3}$$

21. Show that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect. Also find their point of intersection. **4**

Sol. Given lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7} = \lambda$

and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5} = \mu$

General points on the lines

$\{(3\lambda - 1), (5\lambda - 3), (7\lambda - 5)\}$ and $\{(\mu + 2), (3\mu + 4), (5\mu + 6)\}$ respectively

For intersection of the lines

$$3\lambda - 1 = \mu + 2$$
$$5\lambda - 3 = 3\mu + 4$$
$$7\lambda - 5 = 5\mu + 6$$
$$3\lambda - \mu = 3 \quad ...(i)$$
$$5\lambda - 3\mu = 7 \quad ...(ii)$$
$$7\lambda - 5\mu = 11 \quad ...(iii)$$

From (i) & (ii), $\lambda = \frac{1}{2}$ and $\mu = -\frac{3}{2}$

$\lambda = \frac{1}{2}$ and $\mu = \frac{-3}{2}$ satisfies the equation (iii)

$\therefore$ Lines are intersect each other and point of intersection is

$$\left(\frac{3}{2}-1\right), \left(\frac{5}{2}-3\right), \left(\frac{7}{2}-5\right)$$
$$\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$$

22. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls? Given that

(i) the youngest is a girl.

(ii) atleast one is a girl. **4**

Sol. Sample space {(B, B), (B, G), (G, B), (G, G)}

$\therefore$ $n(S) = 4$

(i) Both are girls $P(A) = \frac{1}{4}$

Youngest is a girls $P(B) = \frac{1}{2}$

$$\therefore \quad P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$
$$= \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

(ii) Atleast one is girl $P(C) = \frac{3}{4}$

$$P\left(\frac{A}{C}\right) = \frac{P(A \cap C)}{P(C)}$$
$$= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

SECTION - C

23. Two schools P and Q want to award their selected students on the values of Discipline, Politeness and Punctuality. The school P wants to ward ₹ x each, ₹ y each and ₹ z each for the three respective values to its 3, 2 and 1 students with a total award money of ₹ 1,000. School Q wants to spend ₹ 1,500 to award its 4, 1 and 3 students on the respective values (by giving the same award money for the three values as before). If the total amount of awards for one prize on each value is ₹ 600, using matrices, find the award money for each value. Apart from the above three values, suggest one more value for awards. **6**

Sol. Given the awards for sincerity, truth fulness and helpfulness are ₹ x, ₹ y and ₹ z respectively.

$$\therefore \quad 3x + 2y + z = 1000$$
$$4x + y + 3z = 1500$$
$$x + y + z = 600$$

Equation can be written in the matrix form

$$\begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 600 \end{bmatrix}$$

Where $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B = \begin{bmatrix} 1000 \\ 1500 \\ 600 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$
$$= 3(1-3) - 2(4-3) + 1(4-1)$$
$$= -5$$

$$\text{Adj } A = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 2 & +1 & -5 \\ +1 & -2 & 5 \\ -3 & 1 & 5 \end{bmatrix}$$

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & 1 & -5 \\ 1 & -2 & 5 \\ -3 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1000 \\ 1500 \\ 600 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 2000+1500-3000 \\ 1000-3000+3000 \\ -3000+1500+3000 \end{bmatrix}$$

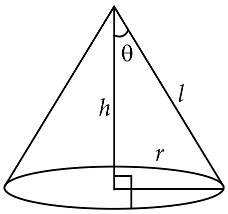
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 500 \\ 1000 \\ 15000 \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$

$\therefore$ x = ₹ 100, y = ₹ 200 and z = ₹ 300

One more value is should be discipline.

24. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\cos^{-1}\frac{1}{\sqrt{3}}$. **6**

Sol.



(Slant height is given $\therefore$ l is constant)

$$\text{Volume of cone } V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi(l^2 - h^2)h$$

$$V = \frac{1}{3}\pi(l^2h - h^3)$$

$$\frac{dV}{dh} = \frac{1}{3}\pi\{l^2 - 3h^2\}$$

$$\text{For maximum volume } \frac{dV}{dh} = 0$$

$$0 = \frac{1}{3}\pi(l^2 - 3h^3)$$

$$l^2 - 3h^2 = 0$$

$$\Rightarrow \quad l = \sqrt{3}h$$

$$r^2 + h^2 - 3h^2 = 0$$

$$r = \sqrt{2}h$$

$$\frac{d^2V}{dh^2} = \frac{1}{3}\pi(-6h) < 0$$

$\therefore$ Volume is maximum when $l = \sqrt{3}h$

$$\cos\theta = \frac{h}{l} = \frac{h}{\sqrt{3}h} = \frac{1}{\sqrt{3}}$$

$$\theta = \cos^{-1}\frac{1}{\sqrt{3}}$$

$\therefore$ Semi vertical angle is $\cos^{-1}\frac{1}{\sqrt{3}}$ **Hence Proved**

25. Evaluate: $\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\cot x}}$.

Sol.

$$I = \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\cot x}}$$

$$I = \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\cot\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}$$

$$= \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}$$

$$= \int_{\pi/6}^{\pi/3}\frac{\sqrt{\cot x}}{1+\sqrt{\cot x}}$$

$$2I = \int_{\pi/6}^{\pi/3}\frac{\sqrt{\cot x}}{1+\sqrt{\cot x}} + \int_{\pi/6}^{\pi/3}\frac{1}{1+\sqrt{\cot x}}$$

$$2I = \int_{\pi/6}^{\pi/3}\frac{1+\sqrt{\cot x}}{1+\sqrt{\cot x}}dx$$

$$= \int_{\pi/6}^{\pi/3}1dx$$

$$2I = [x]_{\pi/6}^{\pi/3}$$

$$2I = \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6}$$

$$I = \frac{\pi}{12}$$

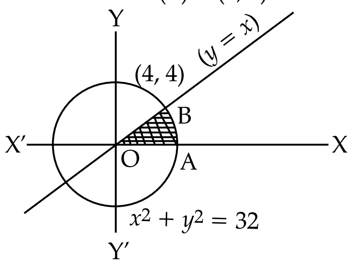
26. Find the area of the region in the first quadrant enclosed by the X-axis, the line $y = x$ and circle $x^2 + y^2 = 32$. **6**

Sol. Given: $x^2 + y^2 = 32$ ...(i)

$y = x$ ...(ii)

from (i) and (ii)

point of intersection (B) is (4, 4)



$\therefore$ Required area = $\int_0^4 x\,dx + \int_4^{4\sqrt{2}}\sqrt{32-x^2}\,dx$

$$= \left[\frac{x^2}{2}\right]_0^4 + \left[\frac{x}{2}\sqrt{32-x^2}+\frac{1}{2}32\sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$= 8-0+\left[0+16.\frac{\pi}{2}-8-16\frac{\pi}{4}\right]$$

$= 8 + 8\pi - 8 - 4\pi$

$= 4\pi$ units^2

* **27. Find the distance between the point (7, 2, 4) and the plane determined by the points A(2, 5, –3), B(–2, –3, 5) and C(5, 3, –3). 6**

28. A dealer in rural area wisher to purchase a number of sewing machines. He has only ₹ 5,760 to invest and has space for at most 20 items for storage. An electronic sewing machine cost him ₹ 360 and a manually operated sewing machine ₹ 240. He can sell an electronic sewing machine at a profit of ₹ 22 and a manually operated sewing machine at a profit of ₹ 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit? Make it as a LPP and solve it graphically. 6

Sol. Let x and y be electronic and manually operated sewing machines Purchased respectively.

$\therefore$ Maximise profit $z = 22x + 18y$

Subject to constraint

$360x + 240y \le 5760$

$x + y \le 20$

$x \ge 0, t \ge 0$

Vertices of feasible region are

(0, 0), (0, 20), (8, 12) and (16, 0)

Profit at (0, 0) $z = 0$

at (0, 20) $z = 0 + 360 =$ ₹ 360

at (8 + 12) $z = 176 + 216 =$ ₹ 392

at (16, 0) $z =$ ₹ 352

To maximum profit electronic $\frac{m}{c} = 8$ and mannual $\frac{m}{c} = 12$.

29. A card from a pack of 52 playing cards is lost. Form the remaining cards of the pack three cards are drawn at random (without replacement) and are found to be all spades. Find the probability of the lost card being a spade. 6

OR

* **From a lot of 15 bulbs which include 5 defectives, a sample of 4 bulbs is drawn one by one with replacement. Find the probability distribution of number of defective bulbs. Hence find the mean of the distribution.**

Sol. Let E_1 = Lost card is a shade

$$\therefore \quad P(E_1) = \frac{13}{52} = \frac{1}{4}$$

E_2 = Lost card is not a spade

$$\therefore \quad P(E_2) = \frac{39}{52} = \frac{3}{4}$$

A = that three spades are drawn without replacement from 51 cards

$$\therefore \quad P\left(\frac{A}{E_1}\right) = \frac{{}^{12}C_3}{{}^{51}C_3} = \frac{12.11.10}{51.50.49} = \frac{1320}{51.50.49}$$

$$P\left(\frac{A}{E_2}\right) = \frac{{}^{13}C_3}{{}^{15}C_3} = \frac{13.12.11}{51.50.49} = \frac{1716}{51.50.49}$$

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1).P\left(\frac{A}{E_1}\right)}{P(E_1).P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{1}{4}\times\frac{1320}{51.50.49}}{\frac{1}{4}\times\frac{1320}{51.50.49}+\frac{3}{4}\times\frac{1716}{51.50.49}}$$

$$= \frac{1320}{1320+5148}$$

$$= \frac{1320}{6468} = \frac{110}{539} = \frac{10}{49}$$

Required probability

$$= \frac{10}{49}$$

Delhi Set II — Code No. 2/1/2

Note : Except for the following questions, all the remaining questions have been asked in previous set.

SECTION - A

9. Evaluate: $\int \cos^{-1}(\sin x)dx$.

Sol. $\int \cos^{-1}(\sin x)dx$

$$= \int \cos^{-1}\left(\cos\left(\frac{\pi}{2}-x\right)\right)dx$$

$$= \int\left(\frac{\pi}{2}-x\right)dx$$

$$= \frac{\pi}{2}x-\frac{x^2}{2}+C$$

10. If vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 3$, $|\vec{b}| = \frac{2}{3}$ and $\vec{a}\times\vec{b}$ is a unit vector, then write the angle between $\vec{a}$ and $\vec{b}$. 1

* *Out of Syllabus*

Sol. $|a| = 3$, $|b| = \frac{2}{3}$, $|\vec{a} \times \vec{b}| = 1$

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$$

$$\sin\theta = \frac{1}{3\times\frac{2}{3}} = \frac{1}{2}$$

$$\therefore \quad \theta = \frac{\pi}{6} \text{ or } 30°$$

SECTION - B

* **19. Prove the following using properties of determinants:**

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$ **4**

20. Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ with respect to $\sin^{-1}(2x\sqrt{1-x^2})$. **4**

Sol. Let

$$u = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$v = \sin^{-1}(2x\sqrt{1-x^2})$$

Put $x = \sin\theta$

$$u = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)$$

$$= \tan^{-1}\tan\theta = \theta$$

$$\therefore \quad u = \sin^{-1} x$$

$$v = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$$

$$= \sin^{-1}\sin 2\theta = 2\theta$$

$$v = 2\sin^{-1} x$$

$$u = \sin^{-1} x, v = 2\sin^{-1} x$$

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}$$

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

$$= \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{2}{\sqrt{1-x^2}}} = \frac{1}{2}$$

* *Out of Syllabus*

21. Solve the following differential equation:

$\text{cosec}\, x \log y \frac{dy}{dx} + x^2y^2 = 0.$ **4**

Sol. $\text{cosec}\, x \log y \frac{dy}{dx} + x^2y^2 = 0$

$$\text{cosec}\, x \log y \frac{dy}{dx} = -x^2y^2$$

$$\frac{\log y}{y^2}dy = \frac{-x^2}{\text{cosec}\, x}dx$$

$$\int y^{-2}\log y\, dy = -\int x^2 \sin x\, dx$$

$$\log y \frac{y^{-1}}{-1} - \int \frac{1}{y}.\frac{y^{-1}}{-1}dy$$

$$= -\left[-x^2\cos x - \int 2x(-\cos x)dx\right]$$

$$-\frac{\log y}{y} + \int y^{-2}dy = +x^2\cos x$$

$$-2\left[x\sin x - \int \sin x\, dx\right]$$

$$\frac{-\log y}{y} - y^{-1} = x^2\cos x - 2x\sin x$$

$$-2\cos x + C$$

$$x^2\cos x - 2x\sin x - 2\cos x + \frac{\log y}{y} + \frac{1}{y} + C = 0$$

* **22. Show that the line $\frac{5-x}{-4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{2y-8}{2} = \frac{z-5}{3}$ are coplanar.** **4**

SECTION - C

28. Evaluate: $\int_0^{\pi} \frac{x\tan x}{\sec x.\text{cosec}\, x}dx$. **6**

Sol.

$$I = \int_0^{\pi} \frac{x\tan x}{\sec x\, \text{cosec}\, x}dx$$

$$I = \int_0^{\pi} \frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)\,\text{cosec}(\pi-x)}dx$$

$$I = \int_0^{\pi} \frac{(\pi-x)\tan x}{\sec x\, \text{cosec}\, x}dx$$

$$I = \int_0^{\pi} \frac{\pi\tan x}{\sec x\, \text{cosec}\, x}dx - \int_0^{\pi} \frac{\pi\tan x}{\sec x\, \text{cosec}\, x}dx$$

$$2I = \pi\int_0^{\pi} \frac{\tan x}{\sec x\, \text{cosec}\, x}dx$$

$$= \pi\int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\sin x\cos x}}dx$$

$$2I = \pi\int_0^{\pi} \sin^2 x\, dx$$

$$= \pi\int_0^{\pi}\left(\frac{1-\cos 2x}{2}\right)dx$$

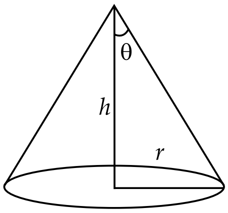
$$2I = \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$2I = \frac{\pi}{2}[\pi - 0]$$

$$\therefore \quad I = \frac{\pi^2}{4}$$

29. Prove that the semi-vertical angle of the right circular cone of given volume and least curved surface area is $\cot^{-1}\sqrt{2}$. **6**

Sol. Let $h_1 r$ and θ be the height, radius and semivertical angle of the cone



GivenL vol. of cone $V = \frac{1}{3}\pi r^2 h$

C.S.A. of cone $(S) = \pi rl$

$$S^2 = \pi^2 r^2 l^2$$

$$Z = \pi^2 r^2 (h^2 + r^2)$$

$$Z = \pi^2 r^2\left(\frac{9V^2}{\pi^2 r^4} + r^2\right)$$

$$Z = \frac{9\pi^2 V^2 r^2}{\pi^2 r^4} + \pi^2 r^4$$

$$= \frac{9V^2}{r^2} + \pi^2 r^4$$

$$\frac{dZ}{dr} = -\frac{18V^2}{r^3} + 4\pi^2 r^3$$

For maxima/Minima $\frac{dZ}{dr} = 0$

$$-\frac{18V^2}{r^3} + 4\pi^2 r^3 = 0$$

$$-18\times\frac{1}{9}\frac{\pi^2 r^4 h^2}{r^3} + 4\pi^2 r^3 = 0$$

$$2\pi^2 r(-h^2 + 2r^2) = 0$$

$$h^2 = 2r^2$$

$$\Rightarrow \quad h = \sqrt{2}r$$

$$\cot\theta = \frac{h}{r} = \sqrt{2}$$

$$\Rightarrow \quad \theta = \cot^{-1}\sqrt{2}$$

$$\frac{d^2Z}{dr^2} = +\frac{54V^2}{r^4} + 12\pi^2 r^3 > 0$$

$\therefore$ C.S.A. is least. **Hence Proved**

Delhi Set III — Code No. 2/1/3

Note: Except for the following questions, all the remaining questions have been asked in previous set.

SECTION - A

9. Evaluate: $\int_0^{\pi/2} e^x(\sin x - \cos x)dx$. **1**

Sol. $\int_0^{\pi/2}(\sin x - \cos x)dx$

Let $f(x) = -\cos x$

$\therefore$ $f'(x) = \sin x$

$$\int e^x(f(x) + f'(x)dx) = e^x f(x) + C$$

$$\therefore \int_0^{\pi/2} e^x(\sin x - \cos x)dx = \left[-e^x\cos x\right]_0^{\pi/2}$$

$$= -0 + 1 = 1$$

10. Write a unit vector in the direction of the sum of the vectors $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 7\hat{k}$. **1**

Sol. Given: $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$

$$\vec{b} = 2\hat{i} + \hat{j} - 7\hat{k}$$

$$\vec{a} + \vec{b} = 4\hat{i} + 3\hat{j} - 12\hat{k}$$

$$|\vec{a} + \vec{b}| = |\sqrt{4^2 + 3^2 + (-12)^2}|$$

$$= |\sqrt{16 + 9 + 144}| = 13 \text{ unit}$$

Unit vector of sum of the vectors

$$= \frac{1}{13}(4\hat{i} + 3\hat{j} - 12\hat{k})$$

$$= \frac{4}{13}\hat{i} + \frac{3}{13}\hat{j} - \frac{12}{13}\hat{k}$$

SECTION - B

*** 19. Using properties of determinants, prove the following:**

$$\begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix} = 1 + x^2 + y^2 + z^2$$ **4**

* *Out of Syllabus*

20. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, when $x \neq 0$. **4**

Sol. Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $V = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan\theta$

$$u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$u = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$u = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$u = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$u = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$u = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1}x$$

$$V = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$V = \sin^{-1}(\sin 2\theta)$$

$$V = 2\theta$$

$$V = 2\tan^{-1}x$$

$$\frac{dV}{dx} = \frac{2}{1+x^2}$$

$$\frac{du}{dx} = \frac{1}{2}.\frac{1}{1+x^2}$$

$$\frac{du}{dV} = \frac{\frac{du}{dx}}{\frac{dV}{dx}}$$

$$= \frac{\frac{1}{2(1+x^2)}}{\frac{2}{(1+x^2)}} = \frac{1}{4}$$

$$\frac{du}{dV} = \frac{1}{4}$$

21. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y + y\cos y}$ given that $y = \frac{\pi}{2}$ when $x = 1$. **4**

Sol.
$$\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y + y\cos y}$$

$$(\sin y + y\cos y)dy = x(2\log x + 1)dx$$

$$\int(\sin y + y\cos y)dy = \int x(2\log x+1)dx$$

$$-\cos y + \int y\cos y\, dy = \frac{x^2}{2} + 2\int x\log x\, dx$$

$$-\cos y + y\sin y - \int \sin y\, dy = \frac{x^2}{2} + 2\left[\frac{x^2}{2}\log x - \int\frac{1}{x}.\frac{x^2}{2}dx\right]$$

$$-\cos y + y\sin y + \cos y = \frac{x^2}{2} + 2\left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right] + C$$

$$y\sin y = \frac{x^2}{2} - \frac{x^2}{2} + x^2\log x + C$$

$$y\sin y = x^2\log x + C$$

$$y = \frac{\pi}{2}, x = 1$$

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1\log 1 + C$$

$$\frac{\pi}{2} = C$$

$\therefore$ Particular solution

$$y\sin y = x^2\log x + \frac{\pi}{2}$$

22. If the lines $\vec{r} = (\hat{i}+\hat{j}-\hat{k}) + \lambda(3\hat{i}-\hat{j})$ and $\vec{r} = (4\hat{i}-\hat{k}) + \mu(2\hat{i}+3\hat{j})$ are intersect then find their point of intersection. **4**

Sol. Two lines are given

$$\vec{r_1} = (\hat{i}+\hat{j}-\hat{k}) + \lambda(3\hat{i}-\hat{j})$$

$$\vec{r_2} = (4\hat{i}-\hat{k}) + \mu(2\hat{i}+3\hat{j})$$

Since, two lines are intersecting

$$\vec{r_1} - \vec{r_2} = \vec{0}$$

$$\vec{r_1} = \vec{r_2}$$

$$\Rightarrow (\hat{i}+\hat{j}-\hat{k}) + \lambda(3\hat{i}-\hat{j}) = (4\hat{i}-\hat{k}) + \mu(2\hat{i}+3\hat{j})$$

$$\Rightarrow (\hat{i}+3\lambda\hat{i}) + (\hat{j}-\lambda\hat{j}) - \hat{k} = (4\hat{i}+2\mu\hat{i}) + 3\mu\hat{j} - \hat{k}$$

$$\Rightarrow \hat{i}(1+3\lambda) + \hat{j}(1-\lambda) - \hat{k} = \hat{i}(4+2\mu) + 3\mu\hat{j} - \hat{k}$$

Now, equating component of $\hat{i}, \hat{j}, \hat{k}$

$$1 + 3\lambda = 4 + 2\mu \quad ...(i)$$

$$1 - \lambda = 3\mu \quad ...(ii)$$

Solving eq (i) and (ii)

$$\mu = 0$$

$$\lambda = 0$$

For intersect point, Put the value of λ and μ either in line $\vec{r_1}$ or line $\vec{r_2}$

For $\vec{r_1}$

$\hat{i}(1+3\lambda) + \hat{j}(1-\lambda) - \hat{k}$

$4\hat{i} - \hat{k}$

For $\vec{r_2}$

$\hat{i}(4+2\mu) + 3\mu\hat{j} - \hat{k}$

$4\hat{i} - \hat{k}$

* *Out of Syllabus*

SECTION - C

28. Evaluate: $\displaystyle\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}$ **6**

Sol.

$$I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)\sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)\cos x \sin x}{\cos^4 x + \sin^4 x} dx$$

$$= \int_0^{\pi/2} \frac{\frac{\pi}{2}\cos x \sin x}{\cos^4 x + \sin^4 x} dx - \int_0^{\pi/2} \frac{x \cos x \sin x}{\sin^4 x + \cos^4 x} dx$$

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\cos x \sin x}{\cos^4 x + \sin^4 x} dx \text{ Devide by } \cos^4 x$$

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\tan x \sec^2 x}{\tan^4 x + 1} dx$$

Put $\tan^2 x = t$

$2 \tan x \sec^2 x \, dx = dt$

$$2I = \frac{\pi}{2}\int_0^{\infty} \frac{dt}{2(t^2 + 1)}$$

$$2I = \frac{\pi}{2} \times \frac{1}{2}[\tan^{-1} t]_0^{\infty}$$

$$2I = \frac{\pi}{4}\left[\frac{\pi}{2} - 0\right]$$

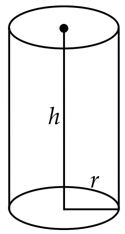
$$I = \frac{\pi^2}{16}$$

29. Of all the closed right circular cylindrical cans of volume 128π cm^3, find the dimensions of the can which has minimum surface area. **6**

Sol. Given vol. of cylinder $= 128\pi$ cm^3

$V = 128\pi$ cm^3

$$\pi r^2 h = 128\pi$$
$$r^2 h = 128$$
$$h = \frac{128}{r^2} \quad \text{...(i)}$$



$$\text{T.S.A. of cylinder} = 2\pi r^2 + 2\pi rh$$

$$S = 2\pi r^2 + 2\pi r \times \frac{128}{r^2}$$

$$S = 2\pi r^2 + \frac{256\pi}{r}$$

$$\frac{dS}{dr} = 4\pi r - \frac{256\pi}{r^2}$$

for minimum surface area

$$0 = 4\pi r - \frac{256\pi}{r^2}$$

$$\frac{256\pi}{r^2} = 4\pi r$$

$$\frac{256\pi}{4\pi} = r^3$$

$$r^3 = 64$$

$\Rightarrow$ $r = 4$ cm

$$\frac{d^2S}{dr^2} = 4\pi + \frac{512\pi}{r^3}$$

$$\left(\frac{d^2S}{dr^2}\right)_{r=4} > 0$$

Hence area is minimum at $r = 4$ cm

$$h = \frac{128}{r^2}$$

$$= \frac{128}{16} = 8 \text{ cm}$$

Radius of cylinder = 4 cm

Height of cylinder = 8 cm

Outside Delhi Set I — Code No. 2/1/1

SECTION - A

1. If $R = [(x, y): x + 2y = 8]$ is a relation of N, write the range of R. **1**

Sol. $R = \{(x, y) \mid x + 2y = 8, x, y \in N\}$

$x + 2y = 8$

$x = 8 - 2y$

$= 8 - 2y > 0 \quad \because x_1\, y \in N$

$= -2y > -8$

$= y < 4$

$\therefore$ Range $= \{1, 2, 3\}$

2. If $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}$, $xy < 1$, then write the value of $x + y + xy$. **1**

Sol.

$$\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}, xy < 1$$

$$\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{x + y}{1 - xy}\right) = \frac{\pi}{4}$$

$$\frac{x+y}{1-xy} = \tan\frac{\pi}{4} = 1$$

$$x + y = 1 - xy$$

$$x + y + xy = 1$$

3. If A is a square matrix such that $A^2 = A$, then write the value of $7A - (I + A)^3$, where I is an identity matrix. **1**

Sol. Given, $A^2 = A$

$$= 7A - (I + A)^3$$
$$= 7A - (I^3 + 3I^2A + 3IA^2 + A^3)$$
$$= 7A - (I + 3A + 3A^2 + A^2.A)$$
$$= 7A - (I + 3A + 3A + A.A)$$
$$= 7A - (I + 6A + A)$$
$$= 7A - I - 7A$$
$$= -I$$

4. If $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$, find the value of $x + y$.

Sol. $$\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$$

$$x - y = -1 \quad ...(i)$$
$$2x - y = 0$$
$\Rightarrow$ $y = 2x$ ...(ii)

from (i), $x - 2x = -1$

$\Rightarrow$ $x = 1$

from (ii), $y = 2 \times 1 = 2$

$$x + y = 1 + 2 = 3$$

5. If $\begin{bmatrix} 3x & 7 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 6 & 4 \end{bmatrix}$, find the value of x. **1**

Sol. $$\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$$

$$12x + 14 = 32 - 42$$
$$12x = -10 - 14$$
$$x = -2$$

6. If $f(x) = \int_0^x t \sin t \, dt$, then write the value of $f'(x)$. **1**

Sol. $$f(x) = \int_0^x t \sin t \, dt$$

$$f(x) = [-t\cos t]_0^x + \int_0^x \cos t \, dt$$

$$= -x\cos x + [\sin t]_0^{2x}$$

$$= -x \cos x + \sin x$$

$$f(x) = \frac{d}{dx}(-x\cos x) + \frac{d}{dx}(\sin x)$$

$$= x \sin x - \cos x + \cos x$$
$$= x \sin x$$

7. Evaluate: $\int_2^4 \frac{x}{x^2+1} dx$.

Sol. $\int_2^4 \frac{x}{x^2+1} dx$ Let $x^2 + 1 = t$

$\int_5^{17} \frac{dt}{2t}$ $2x \, dx = dt$

$x = 2$ then $t = 5$

$x = 4$ then $t = 17$

$$\frac{1}{2}[\log|t|]_5^{17}$$

$$\frac{1}{2}\log\left|\frac{17}{5}\right|$$

8. Find the value of 'p' for which the vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2p\hat{j} + 3\hat{k}$ are parallel. **1**

Sol. Given :

$\because$ Vector $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2p\hat{j} + 3\hat{k}$ are parallel

$\therefore$ $$\frac{3}{1} = \frac{2}{-2p} = \frac{9}{3}$$

$$-6p = 2$$

$\Rightarrow$ $$p = \frac{-2}{6} = \frac{-1}{3}$$

* **9. Find $\vec{a}.(\vec{b} \times \vec{c})$, if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$.** **1**

10. If the Cartesian equations of a line are $\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}$, write the vector equation for the line. **1**

Sol. $$\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}$$

$$\frac{x-3}{-5} = \frac{y-(-4)}{7} = \frac{z-3}{2}$$

$\therefore$ Vector equation of the line

$$= (+3\hat{i} - 4\hat{j} + 3\hat{k}) + \lambda(-5\hat{i} + 7\hat{j} + 2\hat{k})$$

SECTION - B

* **11. If the function $f : R \to R$ be given by $f(x) = x^2 + 2$ and $g : R \to R$ be given by $g(x) = \frac{x}{x-1}$, $x \neq 1$, find $f \circ g$ and $g \circ f$ and hence find $f \circ g$ (2) and $g \circ f(-3)$.** **4**

12. Prove that:

$$\tan^{-1}\left[\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right] = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x, \quad \frac{-1}{\sqrt{2}} \leq x \leq 1.$$

4

* *Out of Syllabus*

OR

If $\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right) = \frac{\pi}{4}$, find the value of x.

Sol. $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$

Let $x = \cos 2\theta$

$$\therefore \tan^{-1}\left[\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right]$$

$$= \tan^{-1}\left[\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right] \quad \text{[Divide by } \cos\theta]$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4}-\theta\right)$$

$$= \frac{\pi}{4}-\theta$$

$$= \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x = \text{R.H.S.}$$

Hence Proved.

OR

$$\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right) = \frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{\frac{x-2}{x-4}+\frac{x+2}{x+4}}{1-\frac{x-2}{x-4}\times\frac{x+2}{x+4}}\right) = \frac{\pi}{4}$$

$$\frac{\frac{x-2}{x-4}+\frac{x+2}{x+4}}{1-\frac{(x-2)(x+2)}{(x-4)(x+4)}} = \tan\frac{\pi}{4}$$

$$\frac{(x-2)(x+4)+(x+2)(x-4)}{(x-4)(x+4)-(x-2)(x+2)} = 1$$

$$\frac{x^2+2x-8+x^2-2x-8}{x^2-16-x^2+4} = 1$$

$$\frac{2x^2-16}{-12} = 1$$

$$2x^2-16 = -12$$

$$x^2 = \frac{4}{2}$$

$$x = \pm\sqrt{2}$$

* **13. Using properties of determinants, prove that:**

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$$ **4**

14. Find the value of $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$ if $x = ae^{\theta}$ (sin θ – cos θ) and $y = ae^{\theta}$ (sin θ + cos θ).

Sol.

$$x = ae^{\theta}(\sin\theta-\cos\theta)$$
$$y = ae^{\theta}(\sin\theta+\cos\theta)$$
$$x = ae^{\theta}(\sin\theta-\cos\theta)$$

$$\frac{dx}{d\theta} = a\left\{(\sin\theta-\cos\theta)\frac{d}{d\theta}e^{\theta}+e^{\theta}\frac{d}{d\theta}(\sin\theta-\cos\theta)\right\}$$
$$= a\{(\sin\theta-\cos\theta)e^{\theta}+e^{\theta}(\cos\theta+\sin\theta)$$
$$= ae^{\theta}(\sin\theta-\cos\theta+\cos\theta+\sin\theta)$$
$$= 2ae^{\theta}\sin\theta$$

$$y = a\,e^{\theta}(\sin\theta+\cos\theta)$$

$$\frac{dy}{d\theta} = a\left\{(\sin\theta+\cos\theta)\frac{d}{d\theta}e^{\theta}+e^{\theta}\frac{d}{d\theta}(\sin\theta+\cos\theta)\right\}$$
$$= a(\sin\theta+\cos\theta)e^{\theta}+e^{\theta}(\cos\theta-\sin\theta)$$
$$= a\,e^{\theta}(\sin\theta+\cos\theta+\cos\theta-\sin\theta)$$
$$= 2ae^{\theta}\cos\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2ae^{\theta}\cos\theta}{2ae^{\theta}\sin\theta} = \cot\theta$$

$$\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = \cot\frac{\pi}{4} = 1$$

15. If $y = Pe^{ax} + Qe^{bx}$, Show that

$$\frac{d^2y}{dx^2}-(a+b)\frac{dy}{dx}+aby = 0.$$ **4**

Sol.

$$y = Pe^{ax} + Q.e^{bx}$$

$$\frac{dy}{dx} = \frac{d}{dx}Pe^{ax}+\frac{d}{dx}Qe^{bx}$$

$$\frac{dy}{dx} = aPe^{ax} + bQe^{bx}$$

$$\frac{d^2y}{dx^2} = a^2Pe^{ax} + b^2Qe^{bx}$$

$$\text{L.H.S.} = \frac{d^2y}{dx^2}-(a+b)\frac{dy}{dx}+aby$$
$$= a^2Pe^{ax} + b^2Qe^{bx} - (a+b)(aPe^{ax} + bQe^{bx}) + ab(Pe^{ax} + Qe^{bx})$$
$$= a^2Pe^{ax} + b^2Qb^{bx} - a^2Pe^{ax} - abQ\,e^{bx} - abPe^{ax} - b^2Qe^{bx} + abPe^{ax} + ab.Qe^{bx}$$
$$= (a^2P - a^2P - abP + abP)e^{ax} + (b^2Q - abQ - b^2Q + abQ)e^{bx}$$
$$= 0\times e^{ax} + 0\times e^{bx}$$
$$= 0 = \text{R.H.S.}$$

Hence Proved.

16. Find the value (s) of x for which $y = [x(x-2)]^2$ is an increasing function. **4**

OR

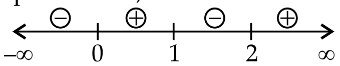
* **Find the equations of the tangent and normal to the curve $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ at the point $(\sqrt{2}a, b)$.**

* *Out of Syllabus*

Sol.

$$y = [x(x-2)]^2$$
$$y = (x^2 - 2x)^2$$
$$\frac{dy}{dx} = \frac{d}{dx}(x^2-2x)^2$$
$$= 2(x^2-2x)\frac{d}{dx}(x^2-2x)$$
$$= 2(x^2 - 2x)\,(2x - 2)$$
$$\frac{dy}{dx} = 4x(x-2)\,(x-1)$$

Critical points are 0, 1 and 2



$$f'(3) = 4 \times 3\,(1)(2) > 0$$
$$f'(1.5) = 4 \times 1.5\,(-0.5)\,(0.5) < 0$$
$$f'(0.5) = 4 \times 0.5\,(-1.5)\,(-0.5) > 0$$
$$f'(-1) = 4\,(-1)\,(-3\,(-2)$$

$\therefore$ function is an increasing in

$[0, 1] \cup [2, \infty]$

17. Evaluate: $\displaystyle\int_0^{\pi} \frac{4x\sin x}{1+\cos^2 x}dx$. **4**

OR

Evaluate: $\displaystyle\int \frac{x+2}{\sqrt{x^2+5x+6}}dx$

Sol.

$$I = \int_0^{\pi} \frac{4x\sin x}{1+\cos^2 x}dx$$
$$I = \int_0^{\pi} \frac{4(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx$$
$$\int_0^a f(x) = \int_0^a f(a-x)dx$$
$$= \int_0^{\pi} \frac{4(\pi - x)\sin x}{1+\cos^2 x}dx$$
$$I = \int_0^{\pi} \frac{4\pi\sin x}{1+\cos^2 x}dx - \int_0^{\pi} \frac{4x\sin x}{1+\cos^2 x}dx$$
$$2I = \int_0^{\pi} \frac{4\pi\sin x}{1+\cos^2 x}dx$$
$$2I = \int_0^{\pi} \frac{4\pi\sin x}{1+\cos^2 x} \qquad \text{Let } \cos x = t$$
$$-\sin x\,dx = dt$$
$$2I = 4\pi\int_1^{-1} \frac{-dt}{1+t^2}$$
$$2I = 4\pi\int_{-1}^{1} \frac{dt}{1+t^2}$$
$$2I = 4\pi\left[\tan^{-1} t\right]_{-1}^{1}$$
$$2I = 4\pi\,[\tan 1 - \tan^{-1}(-1)]$$
$$I = 2\pi\left[\frac{\pi}{4}+\frac{\pi}{4}\right]$$
$$I = \pi^2$$

OR

$$\int \frac{x+2}{\sqrt{x^2+5x+6}}dx$$
$$= \int \frac{\frac{1}{2}(2x+5) - \frac{5}{2} + 2}{\sqrt{x^2+5x+6}}dx$$
$$= \frac{1}{2}\int \frac{2x+5}{\sqrt{x^2+5x+6}}dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+5x+6}}$$
$$= \frac{1}{2}.2\sqrt{x^2+5x+6} - \frac{1}{2}\int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2 + 6 - \frac{25}{4}}}$$
$$= \sqrt{x^2+5x+6} - \frac{1}{2}\int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$
$$= \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|x+\frac{5}{2}+\sqrt{\left(x+\frac{5}{4}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + c$$
$$= \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|x+\frac{5}{2}+\sqrt{x^2+5x+6}\right| + c$$

18. Find the particular solution of the differential equation $\frac{dy}{dx} = 1 + x + y + xy$, given that $y = 0$ when $x = 1$.

Sol.

$$\frac{dy}{dx} = 1 + x + y + xy$$
$$\frac{dy}{dx} = (1+x) + y\,(1+x)$$
$$\frac{dy}{dx} = (1+x)\,(1+y)$$
$$\frac{dy}{1+y} = (1+x)\,dx$$

Integrating both sides

$$\int \frac{dy}{1+y} = \int (1+x)dx$$
$$\log|1+y| = x + \frac{x^2}{2} + c$$
$$y = 0 \text{ when } x = 1$$
$$0 = 1 + \frac{1}{2} + c$$
$$\Rightarrow \qquad c = \frac{-3}{2}$$
$$\log|1+y| = x + \frac{x^2}{2} - \frac{3}{2}$$

19. Solve the differential equation $(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x}$.

Sol. $(1+x^2)\dfrac{dy}{dx}+y = e^{\tan^{-1}x}$

$$\frac{dy}{dx}+\frac{1}{1+x^2}y = \frac{1}{1+x^2}e^{\tan^{-1}x}$$

Differential equation is a form of

$$\frac{dy}{dx}+Py = Q(x)$$

$\therefore$ Integrating factor (I.F.) $= e^{\int P dx}$

$$I.F. = e^{\int\frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$$

Solution of Differential equation

$$y \times I.F. = \int \text{I.F. } Q(x)\, dx$$

$$y \times e^{\tan^{-1}x} = \int e^{\tan^{-1}x}.\frac{e^{\tan^{-1}x}}{1+x^2}dx$$

$$y\, e^{\tan^{-1}x} = \int \frac{e^{2\tan^{-1}x}}{1+x^2}dx$$

$$y\, e^{\tan^{-1}x} = \frac{e^{2\tan^{-1}x}}{2}+c \qquad \begin{bmatrix}\text{Let} \tan^{-1}x = t\\ \dfrac{1}{1+x^2}dx = dt\end{bmatrix}$$

$$y = \frac{1}{2}e^{\tan^{-1}x}+c\,e^{-\tan^{-1}x}$$

* **20. Show that the four points A, B, C and D with position vectors $4\hat{i}+5\hat{j}+\hat{k}$, $-\hat{j}-\hat{k}$, $3\hat{i}+9\hat{j}+4\hat{k}$ and $4(-\hat{i}+\hat{j}+\hat{k})$ respectively are coplanar. 4**

OR

The scalar product of the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ and hence find the unit vector along $\vec{b}+\vec{c}$.

Sol. **OR**

$$\vec{a} = \hat{i}+\hat{j}+\hat{k}$$

$$\vec{b} = 2\hat{i}+4\hat{j}-5\hat{k}$$

$$\vec{c} = \lambda\hat{i}+2\hat{j}+3\hat{k}$$

$$\vec{b}+\vec{c} = (2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$$

$$|\vec{b}+\vec{c}| = |\sqrt{(2+\lambda)^2+6^2+(-2)}|^2$$

$$= |\sqrt{(2+\lambda)^2+40}|$$

$$\vec{a}.(\vec{b}+\vec{c}) = (\hat{i}+\hat{j}+\hat{k}).\left\{\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(21\lambda)^2+40}}\right\}$$

$$1 = \frac{(2+\lambda)+6-2}{\sqrt{(2+\lambda)^2+40}}$$

$$\sqrt{(2+\lambda)^2+40} = \lambda + 6$$

Squaring both side

$$(2+\lambda)^2+40 = (\lambda+6)^2$$

$$4+4\lambda+\lambda^2+40 = \lambda^2+12\lambda+36$$

$$44-36 = 12\lambda-4\lambda$$

$$\frac{8}{8} = 1 = \lambda$$

$$\lambda = 1$$

Unit vector along $(\vec{b}+\vec{c})$ is

$$\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+6^2+(-2)^2}} = \frac{3\hat{i}+6\hat{j}-2\hat{k}}{7}$$

$$= \frac{1}{7}(3\hat{i}+6\hat{j}-2\hat{k})$$

21. A line passes through (2, – 1, 3) and is perpendicular to the lines $\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k})$ and $\vec{r}=(2\hat{i}-\hat{j}-3\hat{k})+\mu(\hat{i}+2\hat{j}+2\hat{k})$. Obtain its equation in vector and Cartesian form. 4

Sol. Given lines $\vec{r} = (\hat{i}+\hat{j}-\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k})$

$$\vec{r} = (2\hat{i}-\hat{j}-3\hat{k})+\mu(\hat{i}+2\hat{j}+2\hat{k})$$

line passes through the point (2, – 1, 3) is

$$\vec{r} = (2\hat{i}-\hat{j}+3\hat{k})+\psi(a\hat{i}+b\hat{j}+c\hat{k})$$

Required line is perpendicular to given lines

$\therefore$ $2a-2b+c = 0$

$a+2b+2c = 0$

$$\frac{a}{-4-2} = \frac{b}{1-4} = \frac{c}{4+2}$$

$$\frac{a}{-6} = \frac{b}{-3} = \frac{c}{6}$$

$$\frac{a}{2} = \frac{b}{1} = \frac{c}{-2}$$

$\therefore$ Requaired line in vector form

$$\vec{r} = (2\hat{i}-\hat{j}+3\hat{k})+\psi(2\hat{i}+\hat{j}-2\hat{k})$$

in cartesian form

$$\frac{x-2}{2} = \frac{y+1}{1} = \frac{z-3}{-2}$$

* **22. An experiment succeeds thrice as often as it fails. Find the probability that in the next five trials, there will be at least 3 successes. 4**

SECTION - C

23. Two schools A and B want to award their selected students on the values of sincerity, truthfulness and helpfulness. The school A wants to award ₹ x each, ₹ y each and ₹ z each for the three respective

* *Out of Syllabus*

values to 3, 2 and 1 students respectively with a total award money of ₹ 1,600. School B wants to spend ₹ 2,300 to award its 4, 1 and 3 students on the respective values (by giving the same award money to the three values as before). If the total amount of award for one prize on each value is ₹ 900, using matrices, find the award money for each value. Apart from these three values, suggest one more value which should be considered for award. **6**

Sol. Let the given awards for sincerity, truth fulness and helpfulness are ₹ x, ₹ y and ₹ z respectively

$$\therefore \quad 3x + 2y + z = 1600$$
$$4x + y + 3z = 2300$$
$$x + y + z = 900$$

Given equations can be written in matrix form

$$\begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix}$$

$$AX = B \qquad ...(1)$$

Where $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix}$

from equ. (1)

$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 3(1-3) - 2(4-3) + 1(4-1)$$
$$= -6 - 2 + 3 = -5$$

$$\text{Adj } A = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{ adj } A = \frac{1}{-5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 2 & 1 & -5 \\ 1 & -2 & 5 \\ -3 & 1 & 5 \end{bmatrix}$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & 1 & -5 \\ 1 & -2 & 5 \\ -3 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3200+2300-4500 \\ 1600-4600+4500 \\ 4800+2300+4500 \end{bmatrix}$$

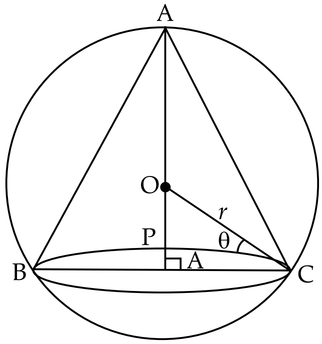
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix}$$

x = ₹ 200, y = ₹ 300 and z = ₹ 400

24. Show that the altitude of the right circular cone of maximum volume that can be described in a sphere of radius r is $\frac{4r}{3}$. Also show that the maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere.

Sol. Let radius of cone be R and height be h respectively

$$OP = r\sin\theta$$
$$OC = R = r\cos\theta$$



$$\therefore \quad \lambda = AP = OA + OP$$
$$= r + r\sin\theta$$
$$= r(1+\sin\theta)$$

Volume of cone

$$V = \frac{1}{3}\pi R^2 h = \frac{1}{3}\pi R^2 r(1+\sin\theta)$$

$$V = \frac{1}{3}\pi r^2\cos^2\theta\, r(1+\sin\theta)$$

$$\frac{dv}{d\theta} = \frac{1}{3}\pi r^3\{\cos^2\theta(\cos\theta) - 2\cos\theta\sin\theta\,(1+\sin\theta)\}$$

$$\frac{dv}{d\theta} = \frac{1}{3}\pi r^3\cos\theta\{\cos^2\theta - 2\sin\theta - 2\sin^2\theta\}$$

$$\frac{dv}{d\theta} = \frac{1}{3}\pi r^3\cos\theta\{1 - 2\sin\theta - 3\sin^2\theta\}$$

$$\frac{dv}{d\theta} = \frac{1}{3}\pi r^3\cos\theta(1+\sin\theta)(1-3\sin\theta)$$

for maximum volume

$$\frac{dv}{d\theta} = 0$$

$$0 = \frac{1}{3}\pi r^3\cos\theta(1+\sin\theta)(1-3\sin\theta)$$

$\therefore \quad \sin\theta = -1, \cos\theta = 0 \text{ and } \sin\theta = \dfrac{1}{3}$

$\theta = \dfrac{-\pi}{2}$ and 0 are not possible

$$\therefore \quad \lambda = r(1+\sin\theta)$$
$$= r\left(1+\frac{1}{3}\right)$$
$$= \frac{4r}{3}\text{unit}$$

$$\frac{dv}{d\theta} = \frac{1}{3}\pi r^3 \cos\theta\,(1+\sin\theta)\,(1-3\sin\theta)$$

$$\frac{d^2v}{d\theta^2} = \frac{1}{3}\pi r^3\{-\sin\theta(1+\sin\theta)\,(1-3\sin\theta) + \cos\theta\,(\cos\theta)\,(1-3\sin\theta) + \cos\theta\,(1+\sin\theta)\,(-3\cos\theta)\}$$

$$\left(\frac{d^2v}{d\theta^2}\right)_{\sin\theta=\frac{1}{3}} = \frac{1}{3}\pi r^3\left\{-\frac{1}{3}\left(1+\frac{1}{3}\right)(1-1) + \frac{2\sqrt{2}}{3}\times\frac{2\sqrt{2}}{3}(1-1) + \frac{2\sqrt{2}}{3}\left(1+\frac{1}{3}\right)\left(-3\times\frac{2\sqrt{2}}{3}\right)\right\}$$

$$\left(\frac{d^2v}{d\theta^2}\right)_{\sin\theta=\frac{1}{3}} = \frac{1}{3}\pi r^3\left\{0+0-\frac{32}{9}\right\} < 0$$

Hence volume is maximum at $\sin\theta = \dfrac{1}{3}$

$$V = \frac{1}{3}\pi r^3\cos^2\theta\,(1+\sin\theta)$$
$$= \frac{1}{3}\pi r^3\left(1-\frac{1}{9}\right)\left(1+\frac{1}{3}\right)$$
$$= \frac{4}{3}\pi r^3 \times \frac{8}{27}$$

25. Evaluate: $\displaystyle\int \frac{1}{\cos^4 x+\sin^4 x}dx.$ **6**

Sol. $\displaystyle\int \frac{1}{\cos^4 x+\sin^4 x}dx$

Divide by $\cos^4 x\, dx$

$$= \int \frac{\sin^4 x\,dx}{1+\tan^4 x}$$
$$= \int \frac{(1+\tan^2 x)\sec^2 x\,dx}{1+\tan^4 t}$$

Let $\quad \tan x = t$

$\sin^2 x\,dx = dt$

$$= \int \frac{1+t^2}{1+t^4}dt$$

Divide by t^2

$$= \int \frac{1+\frac{1}{t^2}}{\frac{1}{t^2}+t^2}dt$$
$$= \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}dt$$

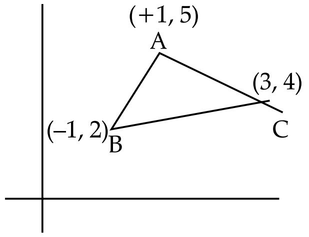
Let $\quad t-\dfrac{1}{t} = u$

$\left(1+\dfrac{1}{t^2}\right)dt = du$

$$= \int \frac{du}{4^2+(\sqrt{2})^2}$$
$$= \frac{1}{\sqrt{2}}\tan^{-1}\left|\frac{u}{\sqrt{2}}\right|+c$$
$$= \frac{1}{\sqrt{2}}\tan^{-1}\left|\frac{t-\frac{1}{t}}{\sqrt{2}}\right|+c$$
$$= \frac{1}{\sqrt{2}}\tan^{-1}\left|\frac{\tan x-\cot x}{\sqrt{2}}\right|+c$$

26. Using integration, find the area of the region bounded by the triangle whose vertices are (–1, 2), (1, 5) and (3, 4). **6**

Sol.



Equation of the line $AB = y-2 = \dfrac{5-2}{1+1}(x+1)$

$$2y-4 = 3x+3$$
$$y = \frac{3x+7}{2}$$

Equation of the line AC

$$y-5 = \frac{4-5}{3-1}(x-1)$$
$$2y-10 = -x+1$$
$$y = \frac{-x+11}{2}$$

Equation of the line BC

$$y-2 = \frac{4-2}{3+1}(x+1)$$
$$2y-4 = x+1$$
$$y = \frac{x+5}{2}$$

Area of ΔABC

$= \int_{-1}^{1}\left(\frac{3x+7}{2}\right)dx + \int_{1}^{3}\left(\frac{-x+11}{2}\right)dx - \int_{-1}^{3}\left(\frac{x+5}{2}\right)dx$

$= \frac{1}{2}\left[\frac{(3x+7)^2}{2\times3}\Big|_{-1}^{1} + \frac{(-x+11)^2}{-2}\Big|_{1}^{3} - \frac{(x+5)^2}{2}\Big|_{-1}^{3}\right]$

$= \frac{1}{4}\left[\frac{100}{3} - \frac{16}{3} - 64 + 100 - 64 + 16\right]$

$= \frac{1}{4}\left[\frac{448}{3} - \frac{400}{3}\right]$

$= 4$ unit^2

* **27. Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$. Also find the distance of the plane obtained above, from the origin.** **6**

OR

* **Find the distance of the point (2, 12, 5) from the point of intersection of the line $\vec{r} = 2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}.(\hat{i} - 2\hat{j} + \hat{k}) = 0$.**

28. A manufacturing company makes two types of teaching aids A and B of Mathematics for class XII. Each type of A required 9 labour hours of fabricating and 1 labour hour for finishing. Each type of B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available per week are 180 and 30 respectively. The company makes a profit of ₹ 80 on each piece of type A and ₹ 120 on each piece of type B. How many pieces of type A and type B should be manufactured per week to get a maximum profit? Make it as an LPP and solve graphically. What is the maximum profit per week? **6**

Sol. Let x and y be the number of teaching aids A and B respectively.

Maximum profit

$$z = 80x + 120y$$

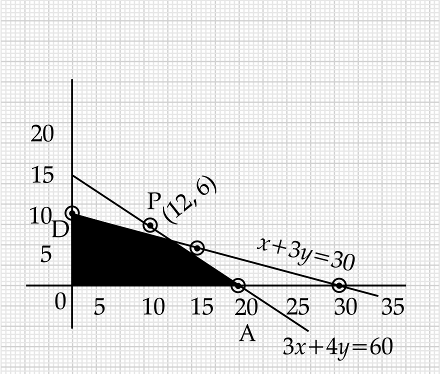
Constaints:

$9x + 12y \le 180$

or $3x + 4y \le 60$

$x + 3y \le 30$

and $x \ge 0, y \ge 0$



$3x + 4y = 60$

x	0	20	10
y	15	0	7.5

$x + 3y = 30$

x	0	30	15
y	10	0	5

The vertices of the feasible region are (0, 0), $A(20, 0)$, $P(12, 6)$ and $D(0, 10)$

$z = 80x + 120y$

at (0, 0) $z = 0$

at (20, 0) $z = 160$

at (12, 6) $z = 960 + 720 = 1680$

at (0, 10) $z = 1200$

Maximum profit at 12 teaching aids types A and 6 teaching aids types B.

Maximum profit = ₹ 1680

29. There are three coins. One is a two-headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the times and third is also a biased coin that comes up tails 40% of the times. One of the three coins is chosen at random and tossed, and it shows heads. What is the probability that it was the two-headed coin? **6**

OR

Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find the probability distribution of the random variable X, and hence find the mean of the distribution.

Sol. Let E_1, E_2 and E_3 be the events of choosing coins of the type two headed bised coxin and showing head 75% of the times and showing head 6% of the times.

$\therefore \quad P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

$P\left(\frac{A}{E_1}\right) = 1, \; P\left(\frac{A}{E_2}\right) = 75\% = \frac{3}{4}$

$P\left(\frac{A}{E_3}\right) = 60\% = \frac{3}{5}$

$P\left(\frac{E_1}{A}\right)$

$$= \frac{P(E_1),P\left(\frac{A}{E_1}\right)}{P(E_1).P\left(\frac{P}{E_1}\right) + P(E_2).P\left(\frac{A}{E_2}\right) + P(E_3)P\left(\frac{A}{E_3}\right)}$$

$$= \frac{\frac{1}{3}\times1}{\frac{1}{3}\times1 + \frac{1}{3}\times\frac{3}{4} + \frac{1}{3}\times\frac{3}{5}}$$

$$= \frac{1}{1 + \frac{3}{4} + \frac{3}{5}}$$

$$= \frac{20}{20+15+12} = \frac{20}{47}$$

* *Out of Syllabus*

Outside Delhi Set II — Code No. 2/1/2

Note: Except for the following questions, all the remaining questions have been asked in previous set.

SECTION - A

9. Evaluate: $\int_e^{e^2} \frac{dx}{x\log x}$

Sol. $\int_e^{e^2} \frac{dx}{x\log x}$

Let
$$\log x = t$$
$$\frac{1}{x}dx = dt$$
$$\int_1^2 \frac{dt}{t} = [\log(t)]_1^2$$
$$= \log 2 - \log 1$$
$$= \log 2$$

10. Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an angle of $\frac{\pi}{4}$ with X-axis, $\frac{\pi}{2}$ with Y-axis and an acute angle θ with Z-axis.

Sol. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{2} + \cos^2\theta = 1$$
$$\frac{1}{2} + 0 + \cos^2\theta = 1$$
$$\cos\theta = \frac{1}{\sqrt{2}}$$
$$\Rightarrow \quad \theta = \frac{\pi}{\sqrt{4}}$$
$$\hat{a} = \frac{1}{\sqrt{2}}\hat{i} + 0\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$$
$$= \frac{1}{\sqrt{2}}(\hat{i} + \hat{k})$$
$$\therefore \quad \vec{a} = 5\sqrt{2}\,\hat{a}$$
$$= 5\sqrt{2} \times \frac{1}{\sqrt{2}}(\hat{i} + \hat{k})$$
$$\vec{a} = 5\hat{i} + 5\hat{k}$$

SECTION - B

20. If $x = a\sin 2t(1 + \cos 2t)$ and $y = b\cos 2t\,(1 - \cos 2t)$, show that at $t = \frac{\pi}{4}$, $\left(\frac{dy}{dx}\right) = \frac{b}{a}$. 4

Sol. $x = a\sin 2t(1 + \cos 2t)$, $y = b\cos 2t(1 - \cos 2t)$

$$y = b\cos 2t(1 - \cos 2t)$$
$$\frac{dy}{dt} = b[\cos 2t(+2\sin 2t) + (1 - \cos 2t)(+2\sin 2t)]$$
$$= b[2\cos 2t\sin 2t - 2\sin 2t - 2\sin 2t\cos 2t]$$
$$\frac{dy}{dt} = -2b\sin 2t$$
$$\frac{dx}{dt} = a[\sin 2t\,(-2\sin 2t) + 2(\cos 2t)(1 + \cos 2t)]$$
$$\frac{dx}{dt} = a[-2\sin^2 2t + 2\cos 2t + 2\cos^2 2t]$$
$$\frac{dx}{dt} = 2a[\cos 4t + \cos 2t]$$
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2b\sin 2t}{2a(\cos 4t + \cos 2t)}$$
$$\frac{dy}{dx} = \frac{b}{a}\left[\frac{-\sin 2t}{\cos 4t + \cos 2t}\right]$$
$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \frac{b}{a}\left[\frac{-\sin\frac{\pi}{2}}{\cos\pi + \cos\frac{\pi}{2}}\right]$$
$$= \frac{b}{a}\left[\frac{-1}{-1+0}\right]$$
$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \frac{b}{a}$$

Hence Proved

21. Find the particular solution of the differential equation $x(1 + y^2)\,dx - y(1 + x^2)\,dy = 0$, given that $y = 1$ when $x = 0$.

Sol.
$$x(1 + y^2)dx - y(1 + x^2)dy = 0$$
$$\frac{x}{1+x^2}dx - \frac{y}{1+y^2}dy = 0$$
$$\int\frac{x}{1+x^2}dx - \int\frac{y}{1+y^2}dy = 0$$
$$\frac{1}{2}\log|1+x^2| - \frac{1}{2}\log|1+y^2| + \log C = 0$$
$$\log\frac{|1+x^2|}{1+y^2}C^2 = 0$$

when $x = 0$, $y = 1$

$$\log\frac{1}{2}C^2 = 0 = \log 1$$

$$\frac{C^2}{2} = 1$$

$\Rightarrow$ $$C^2 = 2$$

$$\log\frac{|2(1+x^2)|}{1+y^2} = 0$$

$\therefore$

$\Rightarrow$ $$2(1 + x^2) = 1 + y^2$$

$$y^2 = 2x^2 + 1$$

22. Find the vector and Cartesian equations of the line passing through the point (2, 1, 3) and perpendicular to the lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x}{-3} = \frac{y}{2} = \frac{z}{5}$. **4**

Sol. Equation of the line passing through the point (2, 1, 3)

$$\frac{x-2}{a} = \frac{y-1}{b} = \frac{z-3}{c}$$

Line is perpendicular to the given lines

$\therefore$ $a + 2b + 3c = 0$

$-3a + 2b + 5c = 0$

$$\frac{a}{10-6} = \frac{b}{-9-5} = \frac{c}{2+6}$$

$$\frac{a}{4} = \frac{b}{-14} = \frac{c}{8}$$

$$\frac{a}{2} = \frac{b}{-7} = \frac{c}{4}$$

$\therefore$ Required line

$$\frac{x-2}{2} = \frac{y-1}{-7} = \frac{z-3}{4}$$

In vector form

$$\vec{r} = (2\hat{i} + \hat{j} + 3\hat{k}) + \lambda(2\hat{i} - 7\hat{j} + 4\hat{k})$$

SECTION - C

28. Evaluate: $\int(\sqrt{\cot x} + \sqrt{\tan x})dx$.

Sol. $\int(\sqrt{\cot x} + \sqrt{\tan x})dx$

Let $\tan x = t^2$

$$\sec^2 x\, dx = 2t\, dt$$

$$(1 + \tan^2 x)dx = 2t\, dt$$

$$dx = \frac{2t}{1+t^4}dt$$

$$= \int\left(\frac{1}{t} + t\right)\frac{2t}{1+t^4}dt$$

$$= \int\frac{2(t^2+1)}{t^4+1}dt \quad \text{Divide by } t^2$$

$$= \int\frac{2\left(1+\frac{1}{t^2}\right)}{t^2+\frac{1}{t^2}}dt$$

$$= \int\frac{2\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+(\sqrt{2})^2}dt$$

$$\left[\begin{array}{l} t-\frac{1}{t} = u \\ \left(1+\frac{1}{t^2}\right)dt = du \end{array}\right]$$

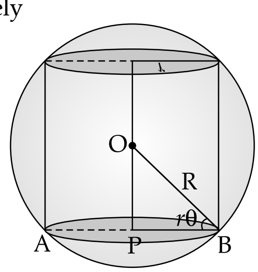
$$= \int\frac{2du}{u^2+(\sqrt{2})^2}$$

$$= \frac{2}{\sqrt{2}}\tan^{-1}\left|\frac{u}{\sqrt{2}}\right| + C$$

$$= \sqrt{2}\tan^{-1}\left|\frac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right| + C$$

29. Prove that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

Sol. Let the radius of the cylinder be r and height be h respectively



Vol. of cylinder $= \pi r^2 h$

$$r = PB = R\cos\theta$$

$$h = 2OP = 2R\sin\theta$$

$$V = \pi R^2\cos^2\theta\, 2R\sin\theta$$

$$V = 2\pi R^3\cos^2\theta.\sin\theta$$

$$V = 2\pi R^3(\cos^2\theta.\cos\theta - 2\cos\theta\sin\theta\sin\theta)$$

$$= 2\pi R^3\cos\theta(\cos^2\theta - 2\sin^2\theta)$$

$$= 2\pi R^3\cos\theta(1 - 3\sin^2\theta)$$

$$\frac{dv}{d\theta} = 0 \text{ (for maxima/minima)}$$

$$2\pi R^3\cos\theta(1 - 3\sin^2\theta) = 0$$

$$1 - 3\sin^2\theta = 0$$

$$\sin\theta = \frac{1}{\sqrt{3}}$$

Height of the cylinder $= h = 2R \sin\theta = \frac{2}{\sqrt{3}}R$

Hence Proved

$$\frac{d^2r}{d\theta^2} = 2\pi R^3[-\sin\theta(1-3\sin^2\theta) + \cos\theta(-6\sin\theta\cos\theta)]$$

$$\left(\frac{d^2r}{d\theta^2}\right)_{\theta=\sin^{-1}\frac{1}{3}} < 0$$

Hence volume is maximum

$$V = 2\pi R^3\cos^2\theta\sin\theta$$
$$= 2\pi R^3(1-\sin^2\theta)\sin\theta$$
$$= 2\pi R^3\left(1-\frac{1}{3}\right)\left(\frac{1}{\sqrt{3}}\right)$$
$$= \frac{4\pi R^3}{3\sqrt{3}} \text{ unit}^3$$

Outside Delhi Set III — Code No. 2/1/3

Note : Except for the following questions, all the remaining questions have been asked in previous set.

SECTION - A

9. If $\int_0^a \frac{1}{4+x^2} = \frac{\pi}{8}$ find the value of a. **1**

Sol.
$$\int_0^a \frac{1}{4+x^2}dx = \frac{\pi}{8}$$
$$\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^a = \frac{\pi}{8}$$
$$\frac{1}{2}\tan^{-1}\frac{a}{2} - 0 = \frac{\pi}{8}$$
$$\tan^{-1}\frac{a}{2} = \frac{\pi}{4}$$
$$\frac{a}{2} = \tan\frac{\pi}{4}$$
$$\therefore \quad a = 2$$

10. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $|\vec{a}+\vec{b}| = 13$ and $|\vec{a}| = 5$, find the value of $|\vec{b}|$.

Sol. Given $\vec{a} \perp \vec{b}$

$$\therefore \quad \vec{a}.\vec{b} = 0$$
$$(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 13^2$$
$$\vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = 13^2$$
$$|a|^2 + 0 + 0 + |b|^2 = 169$$
$$25 + |b|^2 = 169$$
$$|b| = \sqrt{169-25} = 12 \text{ unit}$$

SECTION - B

19. Using properties of determinants, prove that:

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc + bc + ca + ab$$ **4**

Sol.
$$\Delta = \begin{vmatrix} 17a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

$R_1 \to \frac{R_1}{a}, R_2 \to \frac{R_2}{b}$ and $R_3 \to \frac{R_3}{c}$

$$\Delta = abc\begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$\Delta = abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\Delta = (abc + bc + ca + ab)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$$\Delta = (abc + bc + ca)(1-0)$$
$$\Delta = abc + bc + ca = \text{R.H.S.}$$ **Hence Proved**

20. If $x = \cos t(3 - 2\cos^2 t)$ and $y = \sin t(3 - 2\sin^2 t,)$ find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$. **4**

Sol. $x = \cos t(3 - 2\cos^2 t)$

and $y = \sin t(3 - 2\sin^2 t)$

$\therefore$ $x = (3\cos t - 2\cos^3 t)$

and $y = (3\sin t - 2\sin^3 t)$

$$y = 3\sin t - 2\sin^3 t$$

$$\frac{dy}{dt} = 3\cos t - 6\sin^2 t\cos t$$

$$= 3\cos t(1 - 2\sin^2 t)$$

$$= 3\cos t\cos 2t$$

$$x = 3\cos t - 2\cos^3 t$$

$$\frac{dx}{dt} = -3\sin t + 6\cos^2 t\sin t$$

$$= -3\sin t + (1 - 2\cos^2 t)$$

$$= 3\sin t\cos 2t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{3\cos t\cos 2t}{3\sin t\cos 2t}$$

$$\left(\frac{dy}{dx}\right)_{\frac{\pi}{4}} = \cot\frac{\pi}{4} = 1$$

21. Find the particular solution of the differential equation $\left(\frac{dy}{dx}\right) = 3x + 4y$, given that $y = 0$ when $x = 0$. **4**

Sol.

$$\frac{dy}{dx} = 3x + 4y$$

$$\frac{dy}{dx} = 3x$$

Equation in the form of $\frac{dy}{dx} + Py = Q(x)$

$\therefore$ Integrating factor

$$I.F. = e^{\int Pdx} = e^{\int -4dx}$$

$$I.F. = e^{-4x}$$

Solution of the differential equation

$$y + I.F. = \int I.F.\times Q(x)dx$$

$$e^{-4x}y = \int e^{-4x}\times 3x\,dx$$

$$ye^{-4x} = 3\left[\frac{xe^{-4x}}{-4} - \int\frac{e^{-4x}}{-4}dx\right]$$

$$ye^{-4x} = 3\left[-\frac{xe^{-4x}}{4} - \frac{e^{-4x}}{16}\right] + C$$

when $y = 0, x = 0$

$$0 = 3\left[-0 - \frac{1}{16}\right] + C$$

$$C = \frac{3}{16}$$

$\therefore$ Solution of differential equation is

$$ye^{-4x} = -\frac{3xe^{-4x}}{4} - \frac{e^{-4x}}{16} + \frac{3}{16}$$

$$16y = -12x - 1 + 3e^{4x}$$

$$12x + 16y + 1 = 3e^{4x}$$

22. Find the value of p, so that the line $l_1 : \frac{1-x}{3} = \frac{7y-14}{p} = \frac{z-3}{2}$ and $l_2 : \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are perpendicular to each other. Also find the equations of a line passing through a point (3, 2, – 4) and parallel to line l_1. **4**

Sol.

$$l_1 = \frac{7y-14}{p} = \frac{z-3}{2} = \frac{1-x}{3}$$

$$= \frac{y-2}{\frac{p}{7}} = \frac{z-3}{2} = \frac{x-1}{-3}$$

$$l_2 = \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$$

$$= \frac{x-1}{\frac{-3p}{7}} = \frac{y-5}{1} = \frac{z-6}{-5}$$

lines are perpendicular to each other

$$\therefore\ \left(\frac{-3p}{7}\right)(-3) + \frac{p}{7}(1) + 2(-5) = 0$$

$$\frac{9p}{7} + \frac{p}{7} - 10 = 0$$

$$\frac{10p}{7} = 10$$

$\Rightarrow$ $p = 7$

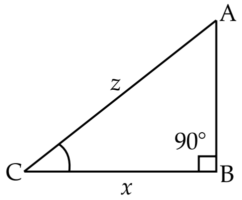
Equation of the line passing through a point (3, 2, –4) and parallel to line l_1 is

$$\frac{x-3}{-3} = \frac{y-2}{1} = \frac{z+4}{2}$$

SECTION - C

28. If the sum of the lengths of the hypotenuse and a side of a right triangle is given, show that the area of the triangle is maximum, when the angle of between them is 60°. **6**

Sol.



Let in ΔABC be right angled at B

and $x + z = \delta$ (Given)

$$\text{Area of } \Delta ABC = \frac{1}{2}BC \times AB$$

$$= \frac{1}{2}xy$$

$$A = \frac{1}{2}x\sqrt{z^2 - x^2}$$

$$A^2 = \frac{1}{4}x^2(z^2 - x^2)$$

$$\frac{dP}{dx} = \frac{1}{4}\frac{d}{dx}(z^2x^2 - x^4)$$

$$(\because A^2 = P)$$

$$\frac{dP}{dx} = \frac{1}{4}(2xz^2 - 4x^3)$$

$$\frac{dP}{dx} = \frac{1}{2}x(z^2 - 4x^2)$$

for maxima/minima

$$\frac{dP}{dx} = 0$$

$$\therefore \quad \frac{1}{2}x(z^2 - 4x^2) = 0$$

$$z - 4x^2 = 0$$

$$z^2 = 4x^2$$

$$z = 2x$$

$$\cos\theta = \frac{BC}{AC} = \frac{x}{2x} = \frac{1}{2}$$

$$\therefore \quad \theta = 60°$$

$$\frac{d^2P}{dx^2} = \frac{1}{4}(2z^2 - 12x^2)$$

$$= \frac{1}{4}(8x^2 - 12x^2)$$

$$= x^2 < 0$$

$\therefore$ Area is maximum when $\angle C = 60°$ **Hence Proved**

29. Evaluate: $\displaystyle\int \frac{1}{\sin^4 x + \sin^2 x\cos^2 x + \cos^4 x}dx.$ **6**

Sol. $\displaystyle\int \frac{1}{\sin^4 x + \sin^2 x\cos^2 x + \cos^4 x}dx$

Divide by $\cos^4 x$

$$= \int \frac{\frac{1}{\cos^4 x dx}}{\frac{\sin 4x}{\cos 4x} + \frac{\sin^2 x\cos^2 x}{\cos^4 x} + \frac{\cos^4 x}{\cos^4 x}}$$

$$= \int \frac{\sec^4 x dx}{\tan^4 x + \tan^2 x + 1}$$

$$= \int \frac{(1+\tan^2 x)\sec^2 x}{\tan^4 x + \tan^2 x + 1}dx$$

let $\tan x = t$
$\sec^2 x\, dx = dt$

$$= \int \frac{1+t^2}{1^4 + t^2 + 1}dt$$

Divide by t^2

$$= \int \frac{1+\frac{1}{t^2}}{t^2 + \frac{1}{t^2} + 1}dt$$

$$= \int \frac{1+\frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 3}dt$$

let $t - \frac{1}{t} = u \Rightarrow \left(1 + \frac{1}{t^2}\right)dt = du$

$$= \int \frac{du}{u^2 + (\sqrt{3})^2}$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{4}{\sqrt{3}} + C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left|\frac{(\tan x - \cot x)}{\sqrt{3}}\right| + C$$

❑❑