ISC Solved Paper 2018

Class-XII

Mathematics

(Maximum Marks : 80)

(Time allowed : Three hours)

*Condidates are allowed an additional **15 minutes** for **only** reading the paper.*
*They must **NOT** start writing during this time.*

Candidates are required to attempt all questions from Section A and all questions EITHER from Section B OR Section C

Section A : Internal choice has been provided in three questions of four marks each and two question of six marks each.

Section B : Internal choice has been provided in two questions of four marks each.

Section C : Internal choice has been provided in two questions of four marks each.

All working, including rough work, should be done on the same sheet as, and adjacent to the rest of the answer.

The intended marks for questions or parts of questions are given in brackets [ ].

Mathematical tables and graph papers are provided.

SECTION - A **[60 Marks]**

1.(i) The binary operation $*: R \times R \to R$ is defined as $a * b = 2a + b$.
Find $(2 * 3) * 4$.

Ans. $a * b = 2a + b, (2 * 3) * 4$? **1**

$$2 * 3 = 2(2) + 3$$
$$= 4 + 3$$
$$2 * 3 = 7$$
$$(2 * 3) * 4 = 7 * 4$$
$$= 2(7) + 4$$
$$= 14 + 4$$
$$= 18$$

(ii) If $A = \begin{pmatrix} 5 & a \\ b & 0 \end{pmatrix}$ and A is symmetric matrix, show that $a = b$

Ans. $$A = \begin{pmatrix} 5 & a \\ b & 0 \end{pmatrix}$$

Given A is symmetric matrix,

$\therefore$ $$A^T = A$$

$$\begin{pmatrix} 5 & b \\ a & 0 \end{pmatrix} = \begin{pmatrix} 5 & a \\ b & 0 \end{pmatrix}$$

$a = b$ Hence proved

(iii) Solve: $3 \tan^{-1} x + cot^{-1} x = \pi$

Ans. $3 \tan^{-1} x + \cot^{-1} x = \pi$

$2 \tan^{-1} x + \tan^{-1} x + \cot^{-1} x = \pi$

[we know that $\tan^{-1} a + \cot^{-1} a = \frac{\pi}{2}$]

$$2 \tan^{-1} x + \frac{\pi}{2} = \pi$$
$$2 \tan^{-1} x = \pi - \frac{\pi}{2}$$
$$2 \tan^{-1} x = \frac{\pi}{2}$$
$$\tan^{-1} x = \frac{\pi}{4}$$
$$x = \tan \frac{\pi}{4}$$
$$x = 1$$

(iv) Without expanding at any stage, find the value of:

$$\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$$

Ans. $$\Delta = \begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$$

$$R_1 \to R_1 + 2R_3$$

$$\Delta = \begin{vmatrix} a+2x & b+2y & c+2z \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$$

$\Delta = 0$ [R_1 and R_2 are identical]

(v) Find the value of constant 'k' so that function $f(x)$ defined as:

$$f(x) = \begin{cases} \dfrac{x^2 - 2x - 3}{x+1}, & x \neq -1 \\ k, & x = -1 \end{cases}$$

is continuous at $x = -1$.

Ans. $$f(x) = \begin{cases} \dfrac{x^2 - 2x - 3}{x+1}, & x \neq -1 \\ k, & x = -1 \end{cases}$$

Given function is continuous at $x = -1$

$$\lim_{x \to -1} f(x) = f(-1)$$

$$\lim_{x \to -1} \frac{x^2 - 2x - 3}{x+1} = k$$

$$\lim_{x \to -1} \frac{x^2 - (3-1)x - 3}{x+1} = k$$

$$\lim_{x \to -1} \frac{x^2 - 3x + x - 3}{x+1} = k$$

$$\lim_{x \to -1} \frac{x(x-3) + 1(x-3)}{(x+1)} = k$$

$$\lim_{x \to -1} \frac{(x-3)(x+1)}{(x+1)} = k$$

$$\lim_{x \to -1} (x-3) = k$$

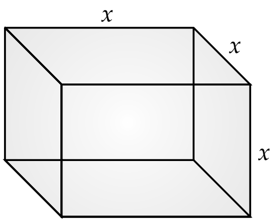
$$-1 - 3 = k$$

$$-4 = k$$

$$k = -4$$

(vi) Find the approximate change in the volume 'V' of a cube of side x metres caused by decreasing the side by 1%.

Ans. Volume of cube = $(\text{side})^3$



$V = x^3$

Given, Decrease in side $= -0.01x$

$$\Delta x = -0.01x$$

$$V = x^3$$

$$\Delta v = \frac{dV}{dx} \times \Delta x$$

$$\Delta v = 3x^2 \times (-0.01x)$$

$$= -0.03x^3$$

Decrease in volume = 3%

(vii) Evaluate : $\displaystyle\int \frac{x^3 + 5x^2 + 4x + 1}{x^2}\, dx$

Ans. $$\int \frac{x^3 + 5x^2 + 4x + 1}{x^2}\, dx$$

$$= \int \left(\frac{x^3}{x^2} + \frac{5x^2}{x^2} + \frac{4x}{x^2} + \frac{1}{x^2} \right) dx$$

$$= \int \left(x + 5 + \frac{4}{x} + x^{-2} \right) dx$$

$$= \int x dx + 5 \int 1 dx + 4 \int \frac{1}{x} dx + \int x^{-2} dx$$

$$= \frac{x^{1+1}}{1+1} + 5x + 4 \log x + \frac{x^{-2+1}}{-2+1} + c$$

$$= \frac{x^2}{2} + 5x + 4 \log x + \frac{x^{-1}}{-1} + c$$

$$= \frac{x^2}{2} + 5x + 4 \log x - \frac{1}{x} + c$$

(viii) Find the differential equation of the family of concentric circles $x^2 + y^2 = a^2$.

Ans. $$x^2 + y^2 = a^2$$

Differentiating *w.r.t* x

$$2x + 2y \frac{dy}{dx} = 0$$

or $$2y \frac{dy}{dx} = -2x$$

or $$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

or $$x + y \frac{dy}{dx} = 0$$

(ix) If A and B are events such that $P(A) = \dfrac{1}{2}$, $P(B) = \dfrac{1}{3}$

and $P(A \cap B) = \frac{1}{4}$, then find:

(a) $P(A / B)$
(b) $P(B / A)$

Ans. $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}, P(A \cap B) = \frac{1}{4}$

(a) $$P(A / B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}$$

(b) $$P(B / A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

(x) In a race, the probabilities of A and B winning the race are $\frac{1}{3}$ and $\frac{1}{6}$, respectively. Find the probability of neither of them winning the race.

Ans. Probability of winning $A = \frac{1}{3}$

$$P(A) = \frac{1}{3}$$

There fore,
Probability of not winning $A = 1 - P(A)$

$$P(\overline{A}) = 1 - P(A) = 1 - \frac{1}{3} = \frac{3-1}{3}$$

$$P(\overline{A}) = \frac{2}{3}$$

Probability of winning, $P(B) = \frac{1}{6}$

There fore,
Probability of not winning B

$$P(\overline{B}) = 1 - P(B) = 1 - \frac{1}{6} = \frac{6-1}{6}$$

$$P(\overline{B}) = \frac{5}{6}$$

Probability of neither of them winning the race

$$= P(\overline{A} \cap \overline{B})$$

$$P(\overline{A} \cap \overline{B}) = P(\overline{A}).P(\overline{B}) = \frac{2}{3} \times \frac{5}{6} = \frac{10}{18} = \frac{5}{9}$$

***2. If the function $f(x) = \sqrt{2x-3}$ is invertible then find its inverse. Hence prove that $(fof^{-1})(x) = x$. [4]**

$$= f\left(\frac{x^2+3}{2}\right)$$

$$= \sqrt{2\left(\frac{x^2+3}{2}\right) - 3}$$

$$= \sqrt{x^2 + \cancel{3} - \cancel{3}}$$

$$= \sqrt{x^2}$$

$(fof^{-1})(x) = x$ **Hence proved.**

3. If $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \pi$, prove that $a + b + c = abc$. **[4]**

Ans. $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \pi$
$\tan^{-1} a + \tan^{-1} b = \pi - \tan^{-1} c$

Using identity $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$.

$$\tan^{-1}\left[\frac{a+b}{1-ab}\right] = \pi - \tan^{-1} c$$

$$\frac{a+b}{1-ab} = \tan(\pi - \tan^{-1} c)$$

$[\because \tan(\pi - \theta) = -\tan\theta]$

$$\frac{a+b}{1-ab} = -\tan[\tan^{-1} c]$$

*Out of Syllabus

$$\frac{a+b}{1-ab} = -c$$

$$\frac{a+b}{1-ab} = \frac{-c}{1}$$

On cross multiplication

$$(a+b)(1) = (-c)(1-ab)$$
$$a+b = -c+abc$$
$$a+b+c = abc$$

Hence proved.

4. Use properties of determinants to solve for x: [4]

$$\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0 \text{ and } x \neq 0.$$

Ans. $\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0$

$$\Delta = \begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & a \\ x+a+b+c & b & x+c \end{vmatrix}$$

Taking $(x+a+b+c)$ common from C_1

$$\Delta = (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & a \\ 1 & b & x+c \end{vmatrix}$$

$R_1 \to R_1 - R_3$ & $R_2 \to R_2 - R_3$

$$\Delta = (x+a+b+c)\begin{vmatrix} 0 & 0 & -x \\ 0 & x & a-x-c \\ 1 & b & x+c \end{vmatrix}$$

On expanding along R_1

$\Delta = (x+a+b+c)\ \{0[(x)(x+c)-(b)(a-x-c)]$
$-(0)[(0)(x+c)-(1)(a-x-c)] + (-x)[(0)(b)-(1)(x)]\}$

$$\Delta = (x+a+b+c)\{0-0-x(0-x)\}$$
$$\Delta = (x+a+b+c)\{x^2\}$$
$$\Delta = x^2(x+a+b+c)$$

Given, $\Delta = 0$

$x^2(x+a+b+c) = 0$

Since $x \neq 0$

So, $x+a+b+c = 0$

or, $x = -(a+b+c)$

5.(a) Show that the function $f(x) = \begin{cases} x^2, & x \leq 1 \\ \dfrac{1}{x}, & x > 1 \end{cases}$ is continuous at $x = 1$ but not differentiable.

OR

(b) Verify Rolle's theorem for the following function: $f(x) = e^{-x}\sin x$ on $[0, \pi]$ [4]

Ans. (a) $\lim_{x\to 1} f(x) = f(1)$

L.H.L

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h)$$
$$= \lim_{h\to 0}(1-h)^2$$
$$= (1-0)^2$$
$$= 1$$

R.H.L

$$= \lim_{h\to 1^+} f(x)$$
$$= \lim_{h\to 0} f(1+h)$$
$$= \lim_{h\to 0}\frac{1}{1+h}$$
$$= \frac{1}{1+0}$$
$$= \frac{1}{1}$$
$$= 1$$
$$f(x) = x^2$$
$$f(1) = (1)^2$$
$$f(1) = 1$$

Here

L.H.L = R.H.L = $f(1)$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$\therefore$ Given function is continuous at $x = 1$

Differentiability at $x = 1$

L.H.D at $(x = 1)$:

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}$$
$$= \lim_{x\to 1}\frac{x^2-1}{x-1}$$
$$= \lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)}$$
$$= \lim_{x\to 1}(x+1)$$
$$= 1+1$$
$$= 2$$

R.H.D at $(x = 1)$

$$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x \to 1} \frac{\frac{1}{x} - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{1 - x}{x(x-1)}$$

$$= \lim_{x \to 1} \frac{-(x-1)}{x(x-1)}$$

$$= \lim_{x \to 1} \frac{-1}{x}$$

$$= -1$$

L.H.D (at $x = 1$) $\neq$ R.H.D (at $x = 1$) so $f(x)$ is not differentiable at $x = 1$.

(b) Since an exponential function (e^{-x}) and sine functions are everywhere continuous and differentiable

$\therefore$ $f(x) = e^{-x}$ sin is Continuous on $[0, \pi]$

and differentiable on $(0, \pi)$

$$f(0) = e^{-x} \sin x = e^{-0} \sin 0$$

$$= \frac{1}{e^0} \times 0 = \frac{1}{1} \times 0 = 0$$

$$f(\pi) = e^{-x} \sin x = e^{-\pi} \sin \pi = \frac{1}{e^{\pi}} \sin x = 0$$

$$f(0) = f(\pi)$$

Hence $f(x)$ satisfies all the three conditions of Rolle's theorem on $[0, \pi]$

$\therefore$ c $[0, \pi]$ such that $f'(c) = 0$

$$f(x) = e^{-x} \sin x$$

$$f'(x) = e^{-x} \cos x + (-e^{-x}) \sin x$$

$$f'(x) = e^{-x} (\cos x - \sin x)$$

$$f'(c) = e^{-c} (\cos c - \sin c)$$

$$0 = e^{-c} (\cos c - \sin c)$$

$$\cos c - \sin c = 0$$

$$\tan c = 1 = \tan \frac{\pi}{4}$$

$$c = \frac{\pi}{4}$$

$$0 < c < \pi$$

Hence, Rolle's theorem verified.

6. If $x = \tan \left(\frac{1}{a} \log y\right)$, prove that $(1 + x^2) \frac{d^2y}{dx^2} + (2x - a) \frac{dy}{dx} = 0$ **[4]**

Ans. If $x = \tan \left(\frac{1}{a} \log y\right)$

$$\tan^{-1} x = \frac{1}{a} \log y$$

Differentiating w.r.t. x

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{a} \frac{d}{dx} \log y$$

$$\frac{1}{1+x^2} = \frac{1}{a} \times \frac{1}{y} \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{ay}{1+x^2} \quad \text{...(i)}$$

Again differentiating w.r.t. x

$$\frac{d^2y}{dx^2} = \frac{(1+x^2)\, a \frac{dy}{dx} - ay \frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(1+x^2)\, a \frac{dy}{dx} - ay\,(0+2x)}{(1+x^2)^2}$$

$$(1 + x^2) \frac{d^2y}{dx^2} = \frac{(1+x^2)\, a \frac{dy}{dx} - 2axy}{(1+x^2)}$$

$$(1 + x^2) \frac{d^2y}{dx^2} = \frac{(1+x^2)\, a \frac{dy}{dx}}{(1+x^2)} - \frac{2axy}{(1+x^2)} = 0$$

$$(1 + x^2) \frac{d^2y}{dx^2} = a \frac{dy}{dx} - \frac{2ax}{(1+x^2)} y$$

$$(1 + x^2) \frac{d^2y}{dx^2} = a \frac{dy}{dx} - 2x \times \left(\frac{ay}{1+x^2}\right)$$

[from eq. (i), $\frac{ay}{1+x^2} = \frac{dy}{dx}$]

$$(1 + x^2) \frac{d^2y}{dx^2} = a \frac{dy}{dx} - 2x \frac{dy}{dx}$$

$$(1 + x^2) \frac{d^2y}{dx^2} = \frac{dy}{dx} (a - 2x)$$

$$(1 + x^2) \frac{d^2y}{dx^2} = -\frac{dy}{dx} (2x - a)$$

$$(1 + x^2) \frac{d^2y}{dx^2} + (2x - a) \frac{dy}{dx} = 0$$

Hence proved.

7. $\int \tan^{-1} \sqrt{x}\, dx$ **[4]**

Ans. $\int \tan^{-1} \sqrt{x}\, dx = I$

$$\int 1 \tan^{-1} \sqrt{x}\, dx \quad \left[\int \underset{\text{I}}{u} \underset{\text{II}}{v}\, dx = u \int v\, dx - \int \left[\frac{d}{dx} u \int v\, dx\right] dx\right]$$

Let $u = \tan^{-1} \sqrt{x}$ and $v = 1$

$$I = \tan^{-1} \sqrt{x} \int 1\, dx - \int \left\{\frac{d}{dx} \tan^{-1} \sqrt{x} \int 1\, dx\right\} dx$$

$$I = \tan^{-1}\sqrt{x}.x - \int \frac{1}{1+\left(\sqrt{x}\right)^2} \times \frac{1}{2\sqrt{x}} x\, dx$$

$$I = x\tan^{-1}\sqrt{x} - \int \frac{x}{2\sqrt{x}(1+x)} dx$$

$$I = x\tan^{-1}\sqrt{x} - \frac{1}{2}\int \frac{\sqrt{x}}{1+x} dx$$

Let $\sqrt{x} = t$

$x = t^2$

$dx = 2t\, dt$

$$I = x\tan^{-1}\sqrt{x} - \frac{1}{2}\int \frac{t}{1+t^2} 2t\, dt$$

$$= x\tan^{-1}\sqrt{x} - \int \frac{t^2}{1+t^2} dt$$

$$= x\tan^{-1}\sqrt{x} - \left[\int 1dt - \int \frac{1}{1+t^2} dt\right]$$

$$= x\tan^{-1}\sqrt{x} - \left[t - \tan^{-1} t\right] + C$$

$$= x\tan^{-1}\sqrt{x} - \sqrt{x} + \tan^{-1}\sqrt{x} + C$$

$$= (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$$

8.(a) Find the points on the curve $y = 4x^3 - 3x + 5$ at which the equation of the tangent is parallel to the x-axis. [4]

OR

(b) Water is dripping out from a conical funnel of semi-verticle angle $\frac{\pi}{4}$ at the uniform rate of 2 cm²/ sec in the surface, through a tiny hole at the vertex of the bottom. When the slant height of the water level is 4 cm, find the rate of decrease of the slant height of the water.

Ans. (a) $y = 4x^3 - 3x + 5$

Differentiating *w.r.t x*

$$\frac{dy}{dx} = 4(3x^2) - 3(1) + 0$$

$$\frac{dy}{dx} = 12x^2 - 3$$

Given equation of the tangent is parallel to the x-axis.

Therefore, $\frac{dy}{dx} = 0$ [$\because$ Slope of the x–axis in O]

$12x^2 - 3 = 0$

$12x^2 = 3$

$$x = \sqrt{\frac{1}{4}}$$

$$x = \pm\frac{1}{2}$$

Put $x = \frac{1}{2}$ in $y = 4x^3 - 3x + 5$

$$y = 4\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right) + 5$$

$$y = 4\left(\frac{1}{8}\right) - \frac{3}{2} + 5$$

$$y = \frac{1}{2} - \frac{3}{2} + 5$$

$$y = \frac{1-3+10}{2}$$

$y = 4$

When $x = \frac{-1}{2}$

$$y = 4\left(\frac{-1}{2}\right)^3 - 3\left(\frac{-1}{2}\right) + 5$$

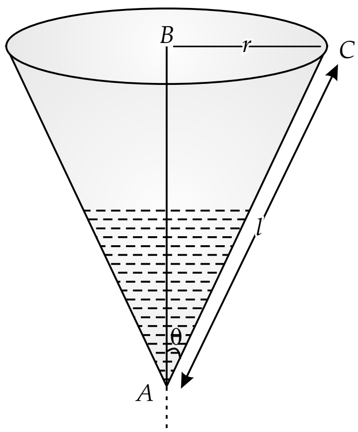
$$y = \frac{-4}{8} + \frac{3}{2} + 5$$

$y = 6$

Hence required points on the curve are $\left(\frac{1}{2}, 4\right)$ and $\left(\frac{-1}{2}, 6\right)$.

OR

(b)



$$\frac{dS}{dt} = -2 \text{ cm}^2/\text{sec}, l = 4, \frac{dl}{dt} = ?, \theta = \frac{\pi}{4}$$

In ΔABC

$$\sin A = \frac{BC}{AC}$$

$$\sin\frac{\pi}{4} = \frac{r}{l}$$

$$\frac{1}{\sqrt{2}} = \frac{r}{l}$$

$$r = \frac{l}{\sqrt{2}} \quad ...(1)$$

Surface Area of con $S = \pi rl$

$$S = \pi\left(\frac{l}{\sqrt{2}}\right) l$$

$$S = \frac{\pi}{\sqrt{2}} l^2$$

Differentiating with respect to t

$$\frac{dS}{dt} = \frac{\pi}{\sqrt{2}} \times 2l\frac{dl}{dt}$$

$$\frac{dS}{dt} = \sqrt{2}\pi l\frac{dl}{dt}$$

$$-2 = \sqrt{2}\pi(4)\frac{dl}{dt}$$

$$\frac{dl}{dt} = \frac{-2}{4\sqrt{2}\pi}$$

$$\frac{dl}{dt} = \frac{-\sqrt{2}}{4\pi}$$

$$= -\frac{1.414}{4 \times 22} \times 7$$

$$\frac{dl}{dt} = -0.11 \text{ cm/sec}$$

Slant height of the water is decreasing at the rate of 0.11 cm/sec.

9.(a) Solve: $\sin x \dfrac{dy}{dx} - y = \sin x.\tan\dfrac{x}{2}$ **[4]**

OR

(b) The population of a town grows at the rate of 10% per year. Using differential equation, find how long will it take for the population to grow 4 times.

Ans. (a) $\sin x \dfrac{dy}{dx} - y = \sin x.\tan\dfrac{x}{2}$

Dividing whole equation by $\sin x$

$$\frac{\sin x}{\sin x}\frac{dy}{dx} - \frac{1}{\sin x}y = \frac{\sin x.\tan\frac{x}{2}}{\sin x}$$

$$\frac{dy}{dx} - \frac{1}{\sin x}y = \tan\frac{x}{2}$$

$$\frac{dy}{dx} - \text{cosec } xy = \tan\frac{x}{2} \quad ...(1)$$

on comparing above mentioned equation with

$$\frac{dy}{dx} + Py = Q$$

$$P = -\text{cosec } x,\ Q = \tan\frac{x}{2}$$

$$I.F = e^{\int P\,dx}$$

$$I.F = e^{\int -\text{cosec } x\,dx}$$

$$I.F = e^{-\int \text{cosec } x\,dx}$$

$$I.F = e^{-\log|\tan\frac{x}{2}|}$$

$$I.F = \frac{1}{e^{\log|\tan\frac{x}{2}|}}$$

$$I.F = \frac{1}{\tan\frac{x}{2}}$$

Thus, the solution of equation (1) is :

$$y \times \frac{1}{\tan\frac{x}{2}} = \int \tan\frac{x}{2} \times \frac{1}{\tan\frac{x}{2}}dx + C$$

$$\frac{y}{\tan\frac{x}{2}} = x + c$$

$$y = (x + c)\tan\frac{x}{2}$$

OR

(b) Let P_0 be the initial population.

Let the population after t year be P.

$$\frac{dP}{dt} = 10\% \text{ of population (Given)}$$

$$\frac{dP}{dt} = \frac{10}{100} \times P = \frac{P}{10}$$

$$\frac{dP}{P} = \frac{dt}{10}$$

$$\int \frac{1}{P}dP = \int \frac{1}{10}dt$$

$$\log|P| = \frac{1}{10}t + C \quad ...(i)$$

When $t = 0$ then $P = P_0$

$$\therefore \quad \log|P_0| = \frac{1}{10} \times 0 + C$$

$$C = \log|P_0| \quad ...(ii)$$

From equation (i) and (ii),

$$\log|P| = \frac{1}{10}t + \log|P_0|$$

$$\log|P| - \log|P_0| = \frac{t}{10}$$

$$\log\left|\frac{P}{P_0}\right| = \frac{t}{10}$$

$$\log 4 = \frac{t}{10}$$

$$0.6020 = \frac{t}{10}$$

$$t = 6.020 \text{ years.}$$

10.(a) Using matrices, solve the following system of equations: [6]

$$\mathbf{2x - 3y + 5z = 11}$$
$$\mathbf{3x + 2y - 4z = -5}$$
$$\mathbf{x + y - 2z = -3}$$

OR

(b) Using elementary transformation, find the inverse of the matrix:

$$\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{3} & \mathbf{1} & \mathbf{1} \end{bmatrix}$$

Ans. (a)

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$\therefore$ $[AA^{-1} = I]$

$$AX = B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B \quad \text{...(i)}$$

Here, $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$|A| = (2)[(2)(-2) - (-4)(1)] - (-3)[(3)(-2) - (1)(-4)] + (5)[(3)(1) - (1)(2)]$$

$$|A| = (2)[-4 + 4] + 3[-6 + 4] + 5[3 - 2]$$
$$= (2)(0) + 3(-2) + 5(1)$$
$$= 0 - 6 + 5$$
$$= -1$$

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$$

$C_{11} = 0,\quad C_{12} = 2,\quad C_{13} = 1$
$C_{21} = -1,\quad C_{22} = -9,\quad C_{23} = -5$
$C_{31} = 2,\quad C_{32} = 23,\quad C_{33} = 13$

$$adj\, A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$adj\, A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T$$

$$adj\, A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$A^{-1} = \frac{adj\, A}{|A|}$$

$$= -1\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$X = A^{-1}B$$

or, $$X = -1\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

or, $$X = -1\begin{bmatrix} (0)(11)+(-1)(-5)+(2)(-3) \\ (2)(11)+(-9)(-5)+(23)(-3) \\ (1)(11)+(-5)(-5)+(13)(-3) \end{bmatrix}$$

or, $$X = -1\begin{bmatrix} 0+5-6 \\ 22+45-69 \\ 11+25-39 \end{bmatrix}$$

or, $$X = -1\begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$$

or, $$X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

or, $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$x = 1, y = 2, z = 3$

OR

(b) Let $$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$$A = IA$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} -1 & -1 & -1 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 + R_3$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - R_3$

$$\begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to \frac{1}{2} R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{-3}{2} & \frac{3}{2} & \frac{-1}{2} \end{bmatrix} A$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{-5}{2} & \frac{3}{2} & \frac{-1}{2} \end{bmatrix} A$$

$R_3 \to (-1) R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

$R_2 \to R_2 - 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ -4 & 3 & -1 \\ \frac{-5}{2} & \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$$

11. *A* speaks truth in 60% of the cases, while *B* in 40% of the cases. In what percent of cases are they likely to contradict each other in stating the same fact? [4]

Ans. Let A and B denote the events 'A speaks the truth' and 'B speaks the truth' respectively.

We have, $P(A) = \frac{60}{100}$

$$P(\bar{A}) = 1 - P(A) = 1 - \frac{60}{100} = \frac{100-60}{100}$$

$$P(\bar{A}) = \frac{40}{100}$$

$$P(B) = \frac{40}{100}$$

$$P(\bar{B}) = 1 - P(B) = 1 - \frac{40}{100} = \frac{100-40}{100}$$

$$P(\bar{B}) = \frac{60}{100}$$

So, required probability

$$= P(A)\,P(\bar{B}) + P(\bar{A})\,P(B)$$

$$= \frac{60}{100} \times \frac{60}{100} + \frac{40}{100} \times \frac{40}{100}$$

$$= \frac{3600}{10000} + \frac{1600}{10000}$$

$$= \frac{3600+1600}{10000}$$

$$= \frac{5200}{10000}$$

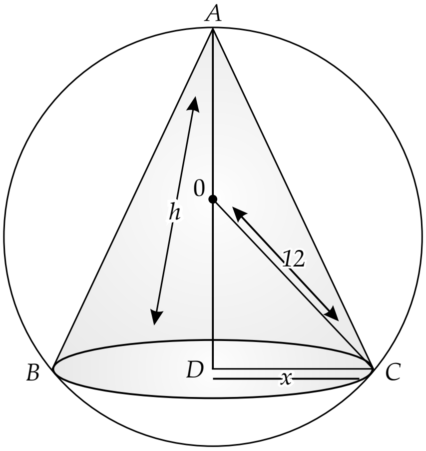
$$= \frac{52}{100}$$

$= 52\%$

Hence they are likely to contradict each other in 52% of the cases in stating the same fact.

12. A cone is inscribed in a sphere of radius 12 cm. If the volume of the cone is maximum, find its height. [6]

Ans.



Let radius of Cone be x cm and its height be h cm

$AD = h$ cm, $\quad OA = OC = 12$ cm

$DC = x$ cm, $\quad OD = h - 12$ cm

Now, In ΔODC

$$OD^2 + DC^2 = OC^2$$
$$(h-12)^2 + x^2 = (12)^2$$
$$x^2 = (12)^2 - (h-12)^2$$
$$x^2 = 144 - h^2 - 144 + 24h$$
$$x^2 = 24h - h^2 \qquad \ldots(i)$$

Volume of cone :

$$V = \frac{1}{3}\pi x^2 h$$

from eq (i)

$$V = \frac{1}{3}\pi(24h - h^2)h$$
$$V = \frac{1}{3}\pi(24h^2 - h^3)$$

Differentiating with respect to h

$$\frac{dV}{dh} = \frac{1}{3}\pi\left[24(2h) - 3h^2\right]$$
$$\frac{dV}{dh} = \frac{\pi}{3}\left[48(h) - 3h^2\right]$$
$$\frac{dV}{dh} = \pi[16h - h^2]$$
$$\frac{dV}{dh} = 0$$
$$\pi[16h - h^2] = 0$$
$$h^2 - 16h = 0$$
$$h[h-16] = 0$$
$$h = 0, 16$$

Height of Cone can not be 0

$$\therefore \quad h = 16$$
$$\frac{dV}{dh} = \pi(16h - h^2)$$

again differentiating with respect to h

$$\frac{d^2V}{dh^2} = \pi[16 - 2h]$$

at $h = 16$

$$\left.\frac{d^2V}{dh^2}\right]_{h=16} = \pi[16 - 2(16)]$$
$$= -16\pi$$
$$\left.\frac{d^2V}{dh^2}\right]_{h=16} < 0\ (-ve)$$

Volume will be maximum at height $h = 16$ cm

13.(a) Evaluate: $\int \frac{x-1}{\sqrt{x^2 - x}}\,dx$

OR

(b) Evaluate: $\int_0^{\pi/2} \frac{\cos^2 x}{1 + \sin x \cos x}\,dx$ **[6]**

Ans. (a) $\int \frac{x-1}{\sqrt{x^2 - x}}dx$

$$x - 1 = A\frac{d}{dx}(x^2 - x) + B$$
$$x - 1 = A(2x - 1) + B$$
$$x - 1 = 2Ax - A + B \qquad \ldots(i)$$

On comparing

$$2A = 1$$
$$A = \frac{1}{2}$$
$$-A + B = -1$$

On putting $A = \frac{1}{2}$

$$\frac{-1}{2} + B = -1$$
$$B = \frac{-1}{2}$$

On putting $A = \frac{1}{2}$ and $B = \frac{-1}{2}$ in eq. (i)

$$x - 1 = \frac{1}{2}(2x-1) - \frac{1}{2}$$

$$\int \frac{x-1}{\sqrt{x^2-x}}dx = \int \frac{\frac{1}{2}(2x-1) - \frac{1}{2}}{\sqrt{x^2-x}}.dx$$

$$= \frac{1}{2}\int \frac{2x-1}{\sqrt{x^2-x}}dx - \frac{1}{2}\int \frac{1}{\sqrt{x^2-x}}dx$$

$$\int \frac{x-1}{\sqrt{x^2-x}}dx = \frac{1}{2}I_1 - \frac{1}{2}I_2 \quad \ldots\text{(ii)}$$

$$I_1 = \int \frac{2x-1}{\sqrt{x^2-x}}dx \qquad \text{Let } x^2 - x = t$$

$$(2x-1)dx = dt$$

$$I_1 = \int \frac{1}{\sqrt{t}}dt$$

$$I_1 = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C_1$$

$$I_1 = 2\sqrt{t} + C_1$$

$$I_1 = 2\sqrt{x^2-x} + C_1 \quad \ldots\text{(iii)}$$

Now

$$I_2 = \int \frac{1}{\sqrt{x^2-x}}dx$$

$$I_2 = \int \frac{1}{\sqrt{x^2-x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

$$I_2 = \int \frac{1}{\sqrt{\left(x-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

$$\text{using : } \int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$I_2 = \log\left|\frac{x-\frac{1}{2}+\sqrt{\left(x-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}{}\right| + C_2$$

$$I_2 = \log\left|\frac{(2x-1)+2\sqrt{x^2-x}}{2}\right| + C_2 \quad \ldots\text{(iv)}$$

$$I_2 = \log\left|(2x-1)+2\sqrt{x^2-x}\right| + C_3 \quad \ldots\text{(iv)}$$

From eq. (iii) and eq. (iv) value of I_1 and I_2 put in eq. (ii) respectively

$$\int \frac{x-1}{\sqrt{x^2-x}}dx = \frac{1}{2}I_1 - \frac{1}{2}I_2$$

$$= \frac{1}{2}\times 2\sqrt{x^2-x} + C_1 - \frac{1}{2}\times\log\left|(2x-1)+2\sqrt{x^2-x}\right| + C_3$$

$$= \sqrt{x^2-x} - \frac{1}{2}\log\left|(2x-1)+2\sqrt{x^2-x}\right| + C$$

OR

(b)

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1+\sin x\cos x}dx \quad \ldots\text{(i)}$$

Using property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\cos x\sin x}dx \quad \ldots\text{(ii)}$$

On adding eq. (i) and eq. (ii)

$$I + I = \int_0^{\frac{\pi}{2}} \left(\frac{\cos^2 x}{1+\sin x\cos x} + \frac{\sin^2 x}{1+\sin x\cos x}\right)dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x+\sin^2 x}{1+\sin x\cos x}dx$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x\cos x}dx$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}+\frac{\sin x\cos x}{\cos^2 x}}dx$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{\sec^2 x+\tan x}dx$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{1+\tan^2 x+\tan x}dx$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{\tan^2 x+\tan x+1}dx$$

let $\tan x = t$

$\sec^2 x dx = dt$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{t^2+t+1}dt$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{t^2 + t + \left(\frac{1}{2}\right)^2 + 1 - \left(\frac{1}{2}\right)^2} dt$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\left(t + \frac{1}{2}\right)^2 + \frac{3}{4}} dt$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dt$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(t + \frac{1}{2}\right)^2} dt$$

$$\left[\text{Using} \int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right]$$

$$I = \frac{1}{2} \times \frac{1}{\frac{\sqrt{3}}{2}} \left[\tan^{-1}\left|\frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right|\right]_0^{\frac{\pi}{2}}$$

$$I = \frac{1}{\sqrt{3}}\left[\tan^{-1}\left|\frac{2t+1}{\sqrt{3}}\right|\right]_0^{\frac{\pi}{2}}$$

$$I = \frac{1}{\sqrt{3}}\left[\tan^{-1}\left|\frac{2\tan x + 1}{\sqrt{3}}\right|\right]_0^{\frac{\pi}{2}}$$

$$I = \frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{2\tan\frac{\pi}{2} + 1}{\sqrt{3}}\right] - \frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{2\tan 0° + 1}{\sqrt{3}}\right]$$

$$I = \frac{1}{\sqrt{3}}\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{\sqrt{3}}\left[\frac{\pi}{2} - \frac{\pi}{6}\right]$$

$$= \frac{1}{\sqrt{3}}\left[\frac{3\pi - \pi}{6}\right]$$

$$= \frac{1}{\sqrt{3}} \times \frac{2\pi}{6}$$

$$I = \frac{\pi}{3\sqrt{3}}$$

14. From a lot of 6 items containing 2 defective items, a sample of 4 items are drawn at random. Let the random variable X denote the number of defective items in the sample. If the sample is drawn without replacement, find:

(a) The probability distribution of X

(b) Mean of X

(c) Variance of X **[6]**

Ans. Let X denotes the number of defective items.

$X = 0, 1, 2$

P(when no defective item found)

$$= \frac{{}^2C_0 \times {}^4C_4}{{}^6C_4} = \frac{1 \times 1}{15} = \frac{1}{15}$$

P(When 1 defective item found in 4 draws)

$$= \frac{{}^2C_1 \times {}^4C_3}{{}^6C_4} = \frac{2 \times 4}{15} = \frac{8}{15}$$

P(When 2 defective item found in 4 draws)

$$= \frac{{}^2C_2 \times {}^4C_2}{{}^6C_4} = \frac{1 \times 6}{15} = \frac{6}{15} = \frac{2}{5}$$

(a) Required Probability distribution is

X	0	1	2
***P*(*X*)**	$\frac{1}{15}$	$\frac{8}{15}$	$\frac{6}{15}$

(b)

$$\text{Mean} = \sum_{i=0}^{2} X_i P(X_i)$$

$$= 0 \times \frac{1}{15} + \frac{1 \times 8}{15} + 2 \times \frac{6}{15}$$

$$= 0 + \frac{8}{15} + \frac{12}{15}$$

$$= \frac{20}{15}$$

$$= \frac{4}{3}$$

(c)

$$\text{Variance} = \sum_{i=0}^{2}\left[x_i^2 P(x_i) - (X_i P(X_i))^2\right]$$

$$= \left[(0)^2 \times \frac{1}{15} + (1)^2 \times \frac{8}{15} + (2)^2 \times \frac{6}{15}\right]$$

$$- \left[0 \times \frac{1}{15} + 1 \times \frac{8}{15} + 2 \times \frac{6}{15}\right]^2$$

$$= \left[0 + \frac{8}{15} + \frac{24}{15}\right] - \left[\frac{4}{3}\right]^2$$

$$= \frac{32}{15} - \frac{16}{9}$$

$$= \frac{96 - 80}{45}$$

$$= \frac{16}{45}$$

SECTION - B **[20 Marks]**

15. (a) Find λ if the scalar projection of $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ is 4 units.

(b) The Cartesian equation of a line is: $2x - 3 = 3y + 1 = 5 - 6z$. Find the vector equation of a line passing through (7, – 5, 0) and parellel to the given line.

(c) Find the equation of the plane through the intersection of the planes $\vec{r}.(\hat{i} + 3\hat{j} - \hat{k}) = 9$ and $\vec{r}.(2\hat{i} - \hat{j} + \hat{k}) = 3$ andpassing through the origin.

[3 × 2]

Ans. (a)

$$\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$$

$$\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$$

$$\frac{\vec{a}.\vec{b}}{|\vec{b}|} = 4 \quad \text{(Given)}$$

$$\frac{(\lambda\hat{i} + \hat{j} + 4\hat{k}).(2\hat{i} + 6\hat{j} + 3\hat{k})}{\sqrt{(2)^2 + (6)^2 + (3)^2}} = 4$$

$$\frac{2\lambda + 6 + 12}{\sqrt{4 + 36 + 9}} = 4$$

$$\frac{2\lambda + 18}{\sqrt{49}} = 4$$

$$\frac{2\lambda + 18}{7} = 4$$

$$2\lambda + 18 = 28$$

$$2\lambda = 28 - 18$$

$$2\lambda = 10$$

$$\lambda = \frac{10}{2}$$

$$\lambda = 5$$

(b) Cartesian equation of line :

$$\frac{2x - 3}{1} = \frac{3y + 1}{1} = \frac{5 - 6z}{1}$$

$$\frac{x - \frac{3}{2}}{\frac{1}{2}} = \frac{y + \frac{1}{3}}{\frac{1}{3}} = \frac{z - \frac{5}{6}}{\frac{-1}{6}}$$

$$\frac{x - \frac{3}{2}}{\frac{1}{2}} = \frac{y - \left(\frac{-1}{3}\right)}{\frac{1}{3}} = \frac{z - \frac{5}{6}}{\frac{-1}{6}}$$

$$a = \frac{1}{2}, b = \frac{1}{3}, c = \frac{-1}{6}$$

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$l = \frac{\frac{1}{2}}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{-1}{6}\right)^2}}$$

$$l = \frac{\frac{1}{2}}{\sqrt{\frac{1}{4} + \frac{1}{9} + \frac{1}{36}}}$$

$$l = \frac{\frac{1}{2}}{\sqrt{\frac{9 + 4 + 1}{36}}}$$

$$l = \frac{\frac{1}{2}}{\sqrt{\frac{14}{36}}}$$

$$l = \frac{\frac{1}{2}}{\sqrt{\frac{7}{18}}}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{\frac{1}{3}}{\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{-1}{6}\right)^2}}$$

$$= \frac{\frac{1}{3}}{\sqrt{\frac{1}{4}+\frac{1}{9}+\frac{1}{36}}}$$

$$= \frac{\frac{1}{3}}{\sqrt{\frac{7}{18}}}$$

Similarly $\quad n = \frac{\frac{-1}{6}}{\sqrt{\frac{7}{18}}}$

Now the vector equation of a line passing through (7,– 5, 0) will be :

$x_1 = 7, y_1 =-5, z_1 = 0$

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

$$\frac{x-7}{\frac{\frac{1}{2}}{\sqrt{\frac{7}{18}}}} = \frac{y+5}{\frac{\frac{1}{3}}{\sqrt{\frac{7}{18}}}} = \frac{z-0}{\frac{\frac{-1}{6}}{\sqrt{\frac{7}{18}}}}$$

$$\frac{2(x-7)}{1} = \frac{3(y+5)}{1} = \frac{-6z}{1}$$

$$2x-14 = 3y-15 = -6z$$

$$\vec{r} = \left(7\hat{i}-5\hat{j}+0\hat{k}\right) + \lambda\left(\frac{1}{2}\hat{i}+\frac{1}{3}\hat{j}-\frac{1}{6}\hat{k}\right)$$

(c) $\quad \vec{r}.(\hat{i}+3\hat{j}-\hat{k}) = 9$

$$\vec{r}.(\hat{i}+3\hat{j}-\hat{k}) - 9 = 0 \quad \ldots\text{(i)}$$

$$\vec{r}.(2\hat{i}-\hat{j}+\hat{k}) = 3$$

$$\vec{r}.(2\hat{i}-\hat{j}+\hat{k}) - 3 = 0 \quad \ldots\text{(ii)}$$

Equation of plane passing through the intersection of the plane

$$[\vec{r}.(\hat{i}+3\hat{j}-\hat{k})-9] + \lambda[\vec{r}.(2\hat{i}-\hat{j}+\hat{k})-3] = 0$$

$$\vec{r}.[(1+2\lambda)\hat{i}+(3-\lambda)\hat{j}+(-1+\lambda)\hat{k}] = 9+3\lambda$$

given that it is passing through origin, such that perpendicular distance of the above plane from origin is zero

i.e., $\left|\frac{d}{\sqrt{a^2+b^2+c^2}}\right| = 0$

$$\left|\frac{9+3\lambda}{(1+2\lambda)^2+(3-\lambda)^2+(-1+\lambda)^2}\right| = 0$$

$$9+3\lambda = 0$$

$$3\lambda = -9$$

$$\lambda = \frac{-9}{3}$$

$$\lambda = -3$$

Equation of plane is

$$\vec{r}.[(1+2(-3))\hat{i}+(3-(-3))\hat{j}+(-1+(-3))\hat{k}] = 9+3(-3)$$

$$\vec{r}.\left[-5\hat{i}+6\hat{j}-4\hat{k}\right] = 9-9$$

$$\vec{r}.\left[-5\hat{i}+6\hat{j}-4\hat{k}\right] = 0$$

16. (a) If *A*, *B*, *C* are three non-collinear points with position vectors $\vec{a}, \vec{b}, \vec{c}$, respectively, then show that the length of the perpendicular from *C* on AB is $\frac{\left|(\vec{a}\times\vec{b})+(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})\right|}{\left|\vec{b}-\vec{a}\right|}$ [4]

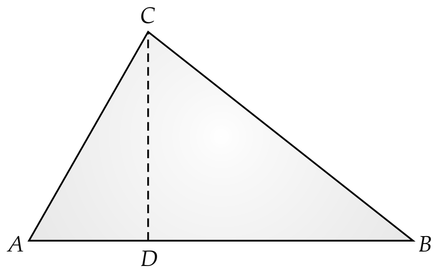
OR

(b) Show that the four points *A, B, C,* and *D* with position vectors $4\hat{i}+5\hat{j}+\hat{k}, -\hat{j}-\hat{k}, 3\hat{i}+9\hat{j}+4\hat{k}$ and $4\left(-\hat{i}+\hat{j}+\hat{k}\right)$ respectively, are coplanar.

Ans. (a) $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OA} = \vec{c}$

We Know that the area of ΔABC

$$= \frac{1}{2} \times AB \times \text{(length perpendicular from } C \text{ on } AB)$$



$$= \frac{1}{2} \times \overrightarrow{AB} \times (\text{length} \perp \text{from } C \text{ on } AB)$$

$$= \frac{1}{2}\left|\overrightarrow{OB}-\overrightarrow{OA}\right| \times (\text{length} \perp \text{from } C \text{ on } AB)$$

$$= \frac{1}{2}\left|\vec{b}-\vec{a}\right| \times (\text{length} \perp \text{from } C \text{ on } AB) \quad \ldots\text{(i)}$$

Also, Area of $\Delta ABC = \frac{1}{2}\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right|$ ...(ii)

From eq (i) and eq (ii)

$\frac{1}{2}\ \left|\vec{b}-\vec{a}\right|$ (length perpendicular from C on AB)

$$= \frac{1}{2}\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}+\vec{a}\right|$$

Length perpendicular from C on AB

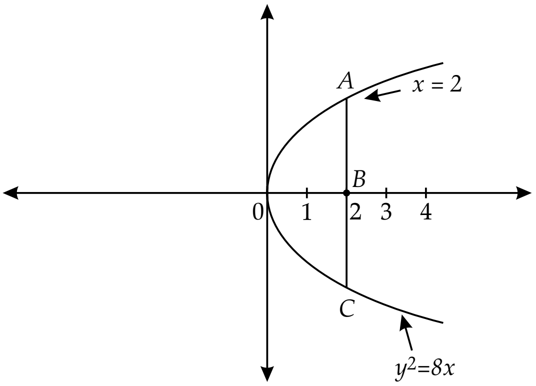
$$= \frac{\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right|}{\left|\vec{b}-\vec{a}\right|}$$

17. (a) Draw a rough sketch of the curve and find the area of the region bounded by curve $y^2 = 8x$ and the line $x = 2$.

OR

(b) Sketch the graph of $y = |x + 4|$. Using integration, find the area of the region bounded by the curve $y = |x + 4|$ and $x = -6$ and $x = 0$. **[4]**

Ans. (a)



Points of intersection at curve $y^2 = 8x$ and line $x = 2$ are (2, 4) and (2 – 4)

To find the area of required region draw a vertical straight length y and width dx

Area of required region $OACO = 2 \times$ area of $OBAO$

$$= 2\int_0^2 y.dx$$

$$= 2\int_0^2 \sqrt{8x}.dx$$

$$= 4\sqrt{2}\int_0^2 (x)^{\frac{1}{2}}.dx$$

$$= 4\sqrt{2}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2$$

$$= 4\sqrt{2}\times\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^2$$

$$= \frac{8}{3}\sqrt{2}\left[(2)^{\frac{3}{2}}-(0)^{\frac{3}{2}}\right]$$

$$= \frac{8}{3}(2)^{\frac{1}{2}}(2)^{\frac{3}{2}}$$

$$= \frac{8}{3}2^{\left(\frac{1}{2}+\frac{3}{2}\right)}$$

$$= \frac{8}{3}\times 2^{\frac{4}{2}}$$

$$= \frac{8}{3}\times(2)^2$$

$$= \frac{8}{3}\times 4$$

$$= \frac{32}{3} \text{ sq unit}$$

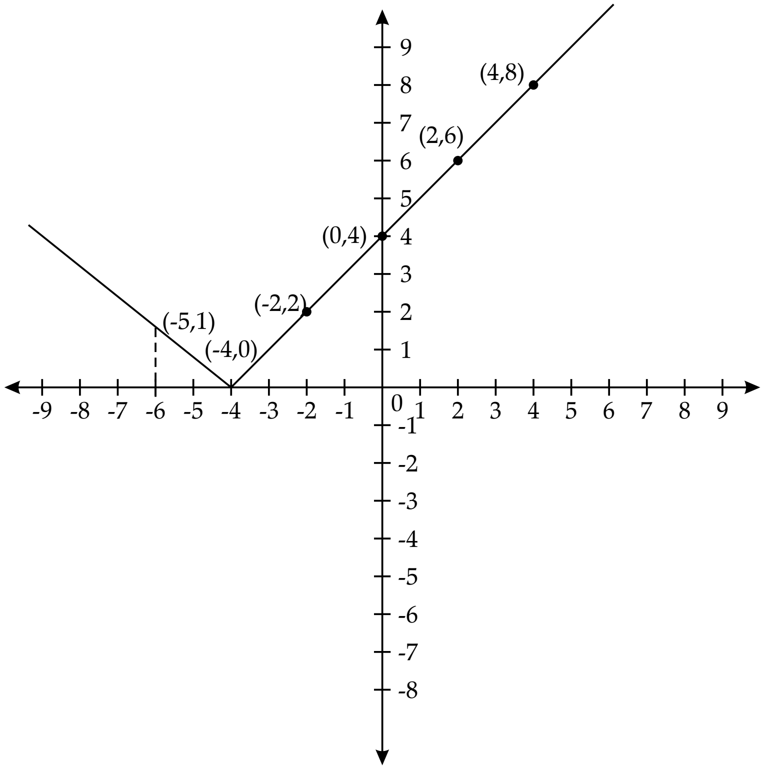
OR

(b) $y = |x + 4|$

$$y = \begin{cases} x+4, & \text{when } x \geq -4 \\ -(x+4), & \text{when } x < -4 \end{cases}$$

$y = |x + 4|$

y	x
4	0
8	4
6	2
2	–2
9	5
1	–5
0	–4



Required area is

$$= \int_{-6}^{0} |x+4|\,dx \;=\; \int_{-6}^{-4} -(x+4)\,dx + \int_{-4}^{0} (x+4)\,dx$$

$$= \left[-\frac{x^2}{2} - 4x\right]_{-6}^{-4} + \left[\frac{x^2}{2} + 4x\right]_{-4}^{0}$$

$$= \left\{\left[-\frac{(-4)^2}{2} - 4(-4)\right] - \left[-\frac{(-6)^2}{2} - 4(-6)\right]\right\} + \left\{\left[\frac{(0)^2}{2} + 4(0)\right] - \left[\frac{(-4)^2}{2} + 4(-4)\right]\right\}$$

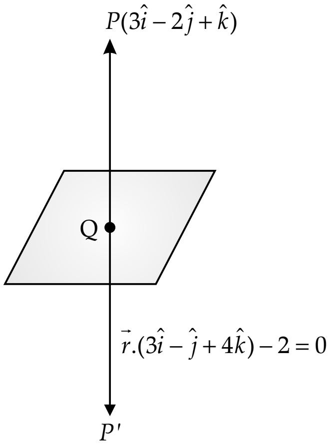
$$= \{[-8+16]-[-18+24]\} + \{[0+0]-[8-16]\}$$

$$= [8-6] + [0-(-8)] = 2 + 8$$

= 10 Sq unit.

18. Find the image of a point having position vector : $3\hat{i} - 2\hat{j} + \hat{k}$ in the plane $\vec{r}.\left(3\hat{i} - \hat{j} + 4\hat{k}\right) = 2$.

Ans.



Equation of line PQ is

$$\vec{r} = (3\hat{i} - 2\hat{j} + \hat{k}) + \lambda(3\hat{i} - \hat{j} + 4\hat{k})$$

$$\vec{r} = (3 + 3\lambda)\hat{i} + (-2 - \lambda)\hat{j} + (1 + 4\lambda)\hat{k}$$

If these are coordinates of point Q, then

$(3)(3 + 3\lambda) + (-1)(-2 - \lambda) + (4)(1 + 4\lambda) - 2 = 0$

$9 + 9\lambda + 2 + \lambda + 4 + 16\lambda - 2 = 0$

$26\lambda + 13 = 0$

$$\lambda = -\frac{1}{2}$$

Coordinated of Q are

$$\left(3 + 3\left(\frac{-1}{2}\right), -2 - \left(\frac{-1}{2}\right), 1 + 4\left(\frac{-1}{2}\right)\right)$$

$$\left(\frac{3}{2}, \frac{-3}{2}, -1\right)$$

Let coordinates of P' be (x, y, z) then :

Here Q is the Mid - Point of P and P'

By using Mid Point formula

$$\frac{x+3}{2} = \frac{3}{2}, \frac{y-2}{2} = \frac{-3}{2}, \frac{z+1}{2} = -1$$

$$x = 0, y = -1, z = -3$$

$$(x, y, z) = (0, -1, -3)$$

SECTION - C [20 Marks]

19. (a) Given the total cost function for x units of a commodity as:

$$C(x) = \frac{1}{3}x^3 + 3x^2 - 16x + 2.$$

Find:

(i) Marginal cost function

(ii) Average cost function

(b) Find the coefficient of correlation from the regression lines:

$x - 2y + 3 = 0$ and $4x - 5y + 1 = 0$.

(c) The average cost function associated with producing and marketing x units of an item is given by $AC = 2x - 11 + \frac{50}{x}$. Find the range of values of the output x, for which AC is increasing.

[3 × 2]

Ans. (a)

$$C(x) = \frac{1}{3}x^3 + 3x^2 - 16x + 2$$

$$C'(x) = \frac{1}{3} \times 3x^2 + 6x - 16$$

$$C'(x) = x^2 + 6x - 16$$

(i) marginal cost $= \frac{d}{dx} C(x)$

$$= C'(x)$$

$$MC = x^2 + 6x - 16$$

(ii)

$$AC = \frac{C(x)}{x}$$

$$= \frac{\frac{1}{3}x^3 + 3x^2 - 16x + 2}{x}$$

$$\therefore \quad AC = \frac{x^2}{3} + 3x - 16 + \frac{2}{x}$$

(b) Let $x - 2y + 3$ is the equation y on x

So $-2y = -x - 3$

$$y = \frac{-x}{-2} - \frac{3}{-2}$$

$$b_{yx} = \frac{-1}{-2} = \frac{1}{2}$$

Let $4x - 5y + 1 = 0$ is the equation x on y

$$4x = 5y - 1$$

$$x = \frac{5y}{4} - \frac{1}{4}$$

$$b_{xy} = \frac{5}{4}$$

Correlation Coefficient $= \sqrt{b_{xy} \times b_{yx}}$

$$= \pm\sqrt{\frac{5}{4} \times \frac{1}{2}}$$

$$= \pm\sqrt{0.625}$$

$$= 0.790 \qquad [b_{yx}/b_{xy} > 0]$$

(c)

$$AC = 2x - 11 + \frac{50}{x}$$

$$\frac{d}{dx}(AC) = 2 - 0 - \frac{50}{x^2}$$

$$\frac{d}{dx}(AC) = 2 - \frac{50}{x^2}$$

For AC to be increasing

$$\frac{d}{dx}(AC) > 0$$

$$2 - \frac{50}{x^2} > 0$$

$$2x^2 - 50 > 0$$

$$x^2 - 25 > 0$$

$$(x - 5)(x + 5) > 0$$

$$x < -5, x > 5$$

Hence, the average cost increases, if the output $x > 5$ i.e., $(5, \infty)$.

20. (a) Find the line of regression of y on x from the following table.

x	1	2	3	4	5
y	7	6	5	4	3

Hence, estimate the value of y when $x = 6$. [4]

OR

(b) From the given data:

Variable	x	y
Mean	6	8
Standard Deviation	4	6

and correlation coefficient: $\frac{2}{3}$. Find:

(i) Regression coefficients b_{yx} and b_{xy}

(ii) Regression line x on y

(iii) Most likely value of x when $y = 14$ [4]

Ans. (a)

X	$X - \overline{X}$ (x)	x^2	Y	$Y - \overline{Y}$ (y)	y^2	xy
1	– 2	4	7	2	4	– 4
2	– 1	1	6	1	1	– 1
3	0	0	5	0	0	0
4	1	1	4	– 1	1	– 1
5	2	4	3	– 2	4	– 4
15	0	10	25	0	10	–10

$$\overline{X} = \frac{15}{5} = 3 \text{ and } \overline{Y} = \frac{25}{5} = 5$$

$$byx = \frac{\Sigma xy - \frac{\Sigma x.\Sigma y}{n}}{\Sigma x^2 - \frac{(\Sigma x)^2}{n}}$$

$$= \frac{\frac{-10-0}{5}}{10 - \frac{(0)^2}{5}} = \frac{-10}{10} = -1$$

Line of regression y on x

$$(y - \overline{y}) = byx\ (x - \overline{x})$$

$$y - 5 = -1\ (x - 3)$$

$$y = -x + 3 + 5$$

$$y = -x + 8$$

Now, when $x = 6$

then,

$$y = -6 + 8$$

$$y = 2$$

OR

(b)

Variable	x	y
Mean	6	8
Standard Deviation	4	6

Correlation Coefficient $= \frac{2}{3}$

(i) $r = \frac{2}{3}$, $\sigma_x = 4$ and $\sigma_y = 6$

$$b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

$$= \frac{2}{3} \times \frac{6}{4} = 1$$

$$b_{xy} = r\frac{\sigma_x}{\sigma_y}$$

$$= \frac{2}{3} \times \frac{4}{6} = \frac{4}{9}$$

(ii) Regression line x on y

$$(x - \overline{x}) = b_{xy}\ (y - \overline{y})$$

$$x - 6 = \frac{4}{9}\ (y - 8)$$

$$x - 6 = \frac{4y}{9} - \frac{32}{9}$$

$$x = \frac{4y}{9} + \frac{22}{9}$$

(iii) When $y = 14$

$$x = \frac{4}{9}\ (14) + \frac{22}{9}$$

$$x = \frac{56}{9} + \frac{22}{9}$$

$$x = \frac{56 + 22}{9} = \frac{78}{9} = \frac{26}{3}$$

21. (a) A product can be manufactured at a total cost $C(x) = \frac{x^2}{100} + 100x + 40$, where x is the number of units produced. The price at which each unit can be sold is given by $P = \left(200 - \frac{x}{400}\right)$. Determine the production level x at which the profit is maximum. What is the price per unit and total profit at the level of production? [4]

OR

(b) A manufacture's marginal cost function is $\frac{500}{\sqrt{2x + 25}}$. Find the cost involved to increase production from 100 units to 300 units. [4]

Ans. Given cost function

$$C(x) = \frac{x^2}{100} + 100\,x + 40$$

$$P = 200 - \frac{x}{400}$$

(i) Revenue function

$$R(x) = P(x) \times x$$

$$R = \left(200 - \frac{x}{400}\right)x$$

$$R = 200\,x - \frac{x^2}{400}$$

(ii) Profit function $= R(x) - C(x)$

$$P(x) = \left(200x - \frac{x^2}{400}\right) - \left(\frac{x^2}{100} + 100x + 40\right)$$

$$= 200\,x - \frac{x^2}{400} - \frac{x^2}{100} - 100\,x - 40$$

$$P(x) = -\frac{x^2}{80} + 100\,x - 40$$

$$\frac{dP}{dx} = \frac{-2x}{80} + 100$$

$$\frac{dP}{dx} = 0$$

$$\frac{-2x}{80} + 100 = 0$$

$$\frac{x}{40} = 100$$

$$x = 4000$$

$$\frac{d^2P}{dx^2} = \frac{-2}{80} = (-\text{ve})$$

Hence, at $x = 4000$ profit is maximum.

$$P(x) = 200 - \frac{x}{400}$$

At $x = 4000$,

Price per unit $= 200 - \frac{4000}{400} = ₹190$

Max. profit $= 100x - \frac{x^2}{80} - 40$

At $x = 4000$,

Max. profit $= 100(4000) - \frac{(4000)^2}{80} - 40 = ₹1,99,960$

OR

(b) $MC = \dfrac{500}{\sqrt{2x+25}}$

Total increased cost when x increases from 100 to 300 units $= C(200) - C(100)$

$$= \int_{100}^{300} MC(x) = \int_{100}^{300} \frac{500}{\sqrt{2x+25}}\,dx = 500\int_{100}^{300} \frac{dx}{\sqrt{2x+25}}$$

$$= 500 \text{ X } \frac{1}{2}\left[\frac{(2x+25)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_{100}^{300}$$

$$= 500\left[\sqrt{2x+25}\,\right]_{100}^{300}$$

$$= 500\left[\sqrt{625} - \sqrt{225}\right]$$

=500 [25-15]

=500[10]

=5000

22. A manufacturing company makes two types of teaching aids *A* and *B* of Mathematics for Class *X*. Each type of *A* requires 9 labour hours for fabricating and 1 labour hour for finishing. Each type of *B* requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available per week are 180 and 30 respectively. The company makes a profit of ₹ 80 on each piece of type *A* and ₹ 120 on each piece of type *B*. How many pieces of type *A* and type *B* should be manufactured per week to get a maximum profit? Formulate this as Linear Programming Problem and solve it. Identify the feasible region from the rough sketch. [6]

Ans.

	$A(x)$	$B(y)$	Total
Fabricating	9	12	180
Finishing	1	3	30

Let no. of pieces of type A and type B manufactured be x and y respectively.

Subject to the constraints :

$$9x + 12y \le 180$$

$$x + 3y \le 30$$

$$x \ge 0$$

$$y \ge 0$$

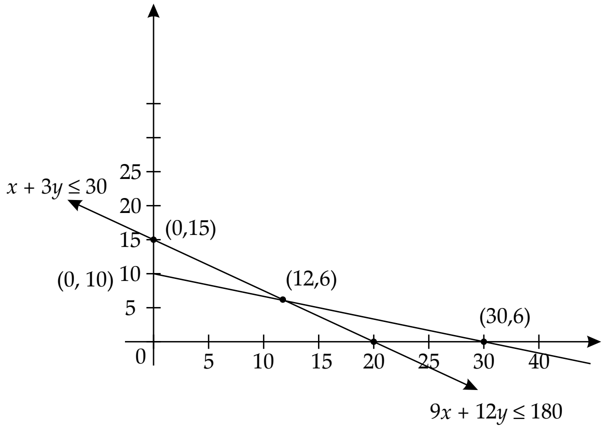
Maximise $Z = 80\,x + 120\,y$

$$9x + 12y = 180$$

x	0	20	12
y	15	0	6

$$x + 3y = 30$$

x	0	30	12
y	10	0	6



The feasible region of the *LPP* is shaded in the fig, the corner points of the feasible region *OAPR* are (0, 0), (20, 0), (12, 6) and (0, 10)

These points have been obtained by solving the cor responding intersecting lines simultaneously The value of the objective function Z at corner points of the feasible region are given is the following table :

Point (x, y)	$\mathbf{Z = 80x + 120y}$
0(0,0)	$Z = 0$
A (20,0)	$Z = 1600$
P (12,6)	$Z = 1680$Max.
B (0,10)	$Z = 1200$

Clearly, Z is maximum at $x = 12$ and $y = 6$

The Maximum value of Z is ₹1680.

Hence the manufacturer should manufactured 12 aids of A and 6 aids of B to obtain maximum profit under the given conditions.

■■